A New Class of Second Order Self-Similar Processes

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December 19, 2003

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Abstract

Self-similarity in discrete second-order stationary processes is defined as a fixed point of a renormalisation operator consisting of aggregation normalised by the variance, rather than by the traditional power-law factor. This broader definition reveals a new class of self-similar processes.

Keywords: self-similarity, renormalisation, fractional noise, stationarity, second-order.

1 Introduction

In second-order stationary processes the second order properties are fully described by a single deterministic quantity, the covariance function. Its normalised form, the correlation function, plays the role of the 'shape parameter' of the process. This paper concerns the definition and correlation structure of self-similar discrete second-order stationary processes. In this context self-similarity is seen as invariance of the correlation function under a suitable kind of renormalisation operation.

[*]Supported by the Australian-European Awards Program, an Australian Government funded Award administered by the Department of Employment, Education, Training and Youth Affairs.

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[‡]Supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant Nos. 25617, 29759, 38396.

We say that the self-similar processes are, by definition, those whose correlation functions are the fixed points of this renormalisation operator, or more precisely, of a family of renormalisation operators forming a semi-group.

The idea of formulating the 'self-similarity' of discrete stationary processes through a fixed point originated in the article of Sinai (1976). In the context of a random field over a d-dimensional lattice, he defined renormalisation as the operation of partitioning the lattice into non-overlapping d-cubes of side m, and replacing the m^d random variables within each cube with their sum, normalised by a suitable power of m. This operator is in many respects very general and is widely used. For example it is equivalent to the definition employed in the review of Cox (1984), which is frequently cited in applications in time series.

In this paper we point out that the a priori choice of power-law normalisation factors is an unnecessary, and unnatural restriction. We examine this in the context of second-order time series, although the restriction to one dimension is not essential. We argue that a normalisation based on the variance of the sum (definition 1) is more natural, and more general, by showing that the set of fixed points one thereby obtains is larger. With power-law normalisation, it is well known (Sinai (1976); Cox (1984)) that the *fractional noise* family contains the only examples of fixed points (typically one speaks of fractional *Gaussian* noise). We show (theorem 1) that the relaxed definition (definition 2) includes this family and in addition at least one other, the *almost periodic* processes. The asymptotic behaviour in this class is oscillatory, in sharp contrast to the smooth power-law behaviour of the fractional noise. Thus, it can only manifest when simplifying constraints which are often imposed, such as assuming that the correlation function decays to zero asymptotically, are not made.

The literature on self-similarity is well developed and already includes generalisations of several kinds. These include self-similarity in higher dimensions Sinai (1976); Major (1981), and non-Gaussian classes both with and without second order moments Sinai (1976); Major (1981); Samorodnitsky and Taqqu (1994). Power-law normalisation factors however are essentially universally used, even for studies of discrete processes, a practice clearly inspired by the known power-law nature of continuous time self-similarity which dominates the literature. Working with approaches and ideas from the continuous world can lead however to intrinsically discrete phenomena being missed. The novelty of our approach is the relaxation of the form of the normalisation step within a classic fixed point definition. As the almost periodic class shows, this can bear fruit in the discrete context.

The question of the significance of the new class is an open one. One way to assess it is through the notion of domains of attraction. It is known that for each member of the fractional Gaussian noise family there exist second order processes which tend to it under repeated renormalisation. Is this also true of the Gaussian incarnations of the almost periodic family? Another open question is the existence of non-Gaussian second order self-similar processes of almost periodic type. In terms of applications, it seems unlikely that such wild processes will be important. If they do appear in nature however, the large magnitude of their correlations should make them easy to detect, and their parameters simple to measure.

2 Renormalisation and Discrete Self-Similarity

Let $\{X(t),\ t \in \mathbb{Z}\}$ denote a discrete time second-order stationary stochastic process. The mean μ and variance $\mathcal{V}$ of such a process are independent of t, and the *autocovariance function*, $\gamma(k) := E[(X(t)-\mu)(X(t+k)-\mu)]$, depends only on the lag k, $k \in \mathbb{Z}$, and $\gamma(k) = \gamma(-k)$. We will assume that the process is non-trivial, that is $\mathcal{V} > 0$. By dividing out this size parameter we obtain the *autocorrelation function* (ACF): $\rho(k) := \frac{\gamma(k)}{\gamma(0)} = \frac{\gamma(k)}{\mathcal{V}}$, which defines the nature of the process.

In addition to these familiar descriptors of second order structure, it turns out to be fruitful to work with an equivalent pair of functions, the *variance time function* and its normalised form, the *correlation time function* (CTF). The variance time function is defined as $\omega(n) = \sum_{k=0}^{n-1}\sum_{i=-k}^{k}\gamma(i) = n\gamma(0) + 2\sum_{i=1}^{n-1} i\gamma(n-i)$, $n = 1, 2, 3\cdots$, while the CTF is just $\phi(n) = \frac{\omega(n)}{\omega(1)} = \frac{\omega(n)}{\mathcal{V}}$. In terms of the original process, $\omega(n)$ is the variance of the sum $\sum_{t=1}^{n} X(t)$. Just as ω can be expressed uniquely in terms of γ via a double sum, γ can be written in terms of ω through a double-differencing operator, defined as:

$$\delta_i^2\{f(i)\}(n) = \begin{cases} f(1) & : n = 0 \\ \frac{1}{2}(f(2) - 2f(1)) & : n = 1 \\ \frac{1}{2}(f(n+1) - 2f(n) + f(n-1)) & : n > 1. \end{cases} \tag{1}$$

It is easy to confirm that these operators are each one-to-one and are inverses of each other, so the autocovariance function and the variance time function are entirely equivalent, and it is obvious that the normalised versions, the ACF and the CTF, are likewise entirely equivalent. Note that $w(1) = \gamma(0) = \mathcal{V}$, so $\phi(1) = \rho(0) = 1$.

Self-similarity relates to invariance with respect to a rescaling operation. In the present context, this decomposes into a rescaling over time, followed by a rescaling of amplitude. For the first of these the standard approach, which we adopt here, is that of *aggregation*. For a fixed $m \geq 1$ the *aggregation of level* m of the original process X is the process $X^{(m)}$ defined as

$$X^{(m)}(t) := \frac{1}{m}\sum_{j=m(t-1)+1}^{mt} X(j). \tag{2}$$

The normalisation factor $1/m$ is conventionally present in definitions of aggregation, as it leaves the mean invariant. As it will be overwritten by the subsequent amplitude rescaling described below, its role is purely incidental and it may be ignored. The γ, ρ, ω, ϕ functions and the variance of the m-aggregated process will be denoted by $\gamma^{(m)}$, $\rho^{(m)}$, $\omega^{(m)}$, $\phi^{(m)}$ and $\mathcal{V}^{(m)}$ respectively.

Understanding the effect of aggregation on the covariance structure amounts to expressing $\gamma^{(m)}$ in terms of γ. Beginning from $\gamma^{(m)}(k) = E[X^{(m)}(0)X^{(m)}(k)] - \mu^2$, the definition, and expanding $X^{(m)}(0)$ and $X^{(m)}(k)$ according to equation (2), we obtain

$$\gamma^{(m)}(k) = \frac{1}{m^2}\left[m\gamma(km) + \sum_{i=1}^{m-1} i(\gamma((k-1)m+i) + \gamma((k+1)m-i))\right]. \tag{3}$$

To express $\omega^{(m)}$ in terms of ω we see from equation (3) that

$$\mathcal{V}^{(m)} = \gamma^{(m)}(0) = \frac{1}{m^2}\left(m\gamma(0) + 2\sum_{i=1}^{m-1} i\gamma(m-i)\right) = \frac{\omega(m)}{m^2}, \tag{4}$$

and so $\mathcal{V}^{(mn)} = \frac{\omega(mn)}{(mn)^2}$. But $\mathcal{V}^{(mn)}$ is the variance of the mn-aggregated process, which is also the n-aggregation of the m-aggregation, and so using equation (4) again we obtain $\mathcal{V}^{(mn)} = \mathcal{V}^{(m)(n)} = \frac{\omega^{(m)}(n)}{n^2}$, from which it follows that

$$\omega^{(m)}(n) = \frac{\omega(mn)}{m^2}. \tag{5}$$

After a m-aggregation rescaling in time, we need to rescale in amplitude. This is performed naturally by dividing by the new variance $\mathcal{V}^{(m)}$, which amounts to examining the effect of aggregation on the correlation structure. Combining the two rescalings by normalising equations (3) and (5) leads to two functional incarnations of the renormalisation operator we consider.

Definition 1 (Renormalisation Operator) *Given a process X with ACF ρ and CTF ϕ, the m-renormalisation consists of a m-aggregation followed by normalisation with $\mathcal{V}^{(m)}$. The renormalised functions are given respectively by*

$$\rho^{(m)}(k) = \left[m\rho(km) + \sum_{i=1}^{m-1} i(\rho((k-1)m+i) + \rho((k+1)m-i)) \right] \Big/ \left[m\rho(0) + 2\sum_{i=1}^{m-1} i\rho(n-i) \right] \tag{6}$$

and

$$\phi^{(m)}(n) = \frac{\phi(mn)}{\phi(m)}. \tag{7}$$

The greater simplicity of the correlation time formulation is apparent.

2.1 Second-Order Self-Similarity

Although different definitions of second-order self-similarity can be found in the literature, they share the common idea of processes which do not change their qualitative statistical behaviour after aggregation. The most natural way to capture this is to directly define a self-similar process as one whose second order structure is a fixed point of the renormalisation operator defined above.

Definition 2 (Second-Order Self-Similarity) *A process is second-order self-similar if $\rho^{(m)} = \rho$, or equivalently $\phi^{(m)} = \phi$, for all $m = 1, 2, 3, \cdots$.*

This definition is not novel, for example it is used by Cox (1984). What is new however is allowing the amplitude rescaling factors $\mathcal{V}^{(m)}$ to take their natural values, which are dependent on the process under study, rather than restricting them according to desired or preconceived notions of their asymptotic form. Cox for example assumes a power-law form for $\mathcal{V}^{(m)}$.

The only examples in the literature of processes obeying this definition are those with the same second order behaviour as fractional Gaussian noise. We call these the *fractional noise* (FN_H) class, defined by $\rho_{\text{FN}}(k) = \delta_i^2\{i^{2H}\}(k)$, or equivalently by the simpler $\phi_{\text{FN}}(n) = n^{2H}$, where $H \in [0,1]$ is the Hurst parameter (Samorodnitsky and Taqqu, 1994). Note that white noise is a member of this family ($H = 1/2$). Exploiting the simplicity of the CTF formulation, we set $\phi^{(m)} = \phi$ in equation (7) to obtain the fixed point equation

$$\phi(nm) = \phi(n)\phi(m), \tag{8}$$

for which it is trivial to show that $\phi_{\mathrm{FN}}(n) = n^{2H}$ is a solution. In Sinai (1976) it is proved that if power-law normalisation factors are selected instead of $\mathcal{V}^{(m)}$, then FN_H is the only possible class of fixed point processes. Since $\mathcal{V}^{(m)} = \mathcal{V}m^{2H-2}$ for FN_H , the two definitions (power-law or variance based normalisation) are equivalent in this important but special case.

In the continuous case it is well known, highly pathological exceptions aside, that the only solutions of equation (8) are power laws. In the discrete case the situation is simpler, but different. It is easily verified that the general solution of equation (8) can be written as

$$\phi(m) = \prod_{i=1}^{s} \phi(p_i)^{r_i}, \quad \text{for each } m = \mathbb{Z}^+, \tag{9}$$

where the p_i are the s distinct prime factors of m, and r_i is the multiplicity of p_i. It is therefore clear that a huge variety of solutions exist which are different, even radically so, from simple power-laws. For example one could even assign the values of ϕ at primes randomly, since at primes there are no constraints imposed by (8) to satisfy, so the function may take any value. In contrast, at non-primes the function is determined by the choices made on its prime factors. It is apparent however that of these fixed point solutions, a great many cannot correspond to valid processes. For example $\phi(n) = n^{\alpha}$ is a solution for all real α, which for $\alpha \in [0, 2]$ is just fractional noise. However $\alpha > 2$ implies that the variance of the sum of m copies of $X(t)$ is greater than $m^2\mathcal{V}$, which is impossible. The missing ingredient is the need for the fixed point solution to also be positive semi-definite.

Recall that a function $f(k)$ defined on $k = 0, 1, 2, \ldots$ is said to be *positive semi-definite* if for any $n \in \mathbb{Z}, n > 0$ and for any real vector $\underline{\mathbf{a}}$ of length n

$$\sum_{1\leq i,j\leq n} a_i f(|i-j|) a_j \geq 0.$$

It can be shown that a necessary and sufficient condition for a function to be the autocovariance function of a process is that it be positive semi-definite (Brockwell and Davis, 1996). In order to have a uniform notation across the two formulations, we introduce the following term.

Definition 3 (Valid functions) *A function γ is said to be a valid (autocovariance or autocorrelation) function if it is positive semi-definite.*
A function ω is said to be a valid (variance time or correlation time) function if $\gamma = \delta^2\{\omega\}$ is positive semi-definite.

The requirement that ϕ be valid imposes strong constraints on the solutions of (8). It can however be very difficult to judge if a given function is positive semi-definite, and therefore the existence of new fixed point solutions which are simultaneously valid processes is far from obvious.

Figure 1 shows three examples of ϕ that satisfy (8), and the corresponding ρ functions calculated as $\rho := \delta^2\{\phi\}$. The first column shows a fractional noise fixed point with $H = 0.6$. The second is a perturbation of $\mathrm{FN}_{0.6}$, where at a small number of primes $\{q_i\}$, the $\phi(q_i)$ have been selected to fall on the curve $n^{1.184}$ rather than $n^{1.2}$. The resulting ϕ is constrained to fall within these two curves (shown in the figure), but we see that the ACF quickly oscillates outside of $[-1, 1]$, betraying its lack of positive semi-definiteness, and the invalidity of the fixed point. The third column will be discussed in the next section.

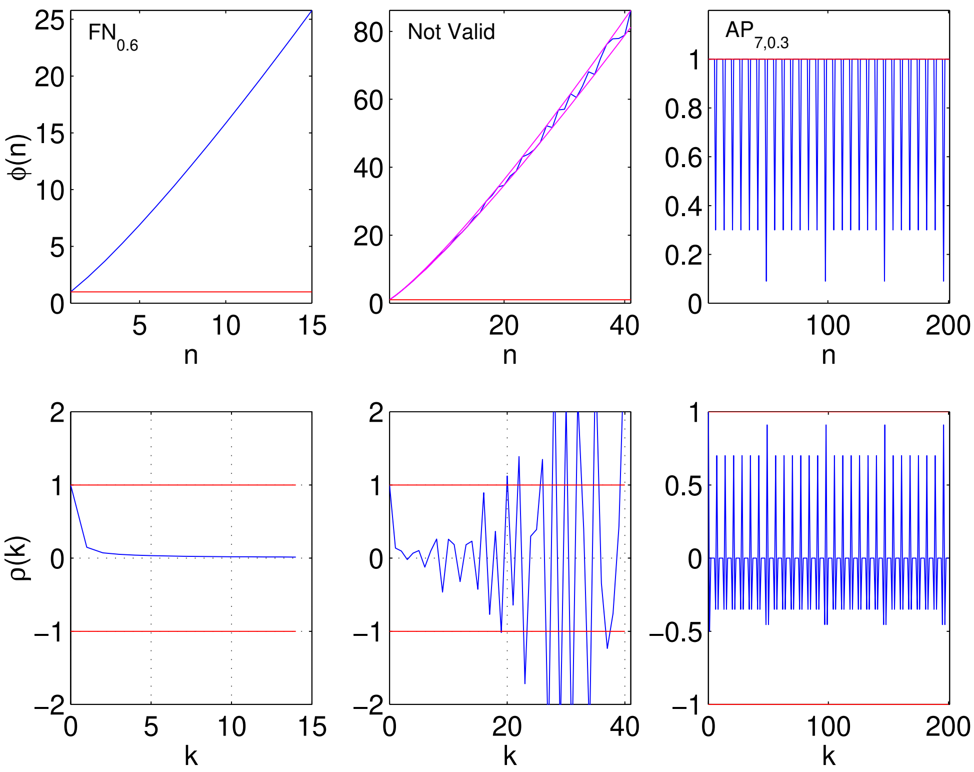

Figure 1: Examples of ϕ functions satisfying (8) (top row) and corresponding ρ functions (bottom). First column: fractional noise FN_H with $H = 0.6$, Second column: a small perturbation of the first column which is not valid, Third column: an almost periodic fixed point $\mathrm{AP}_{q,c}$ with $q = 7$, $c = 0.3$.

3 The Almost Periodic Class

In this section we study the following two parameter class of functions, which are fixed point solutions of (8) through the general prime factor construction of equation (9).

Definition 4 (The almost periodic fixed point family $\mathrm{AP}_{q,c}$) *The two parameter family of fixed points defined by $\phi(1) = 1$, $\phi(p) = 1$ for all primes p except $p = q$, where $\phi(q) = c$, $c \in (0, 1)$, will be called Almost Periodic, and denoted by $AP_{q,c}$.*

A member of the family is exhibited in the right column of figure 1, where its 'almost periodic' nature is readily appreciated. That it is not periodic can be seen from the fact that a new value of the function is reached for the first time whenever n reaches the next higher power of q. Note that both ρ and ϕ oscillate and therefore fail to have limits at infinity.

Each member of this class is a fixed point, but it is not obvious how to show that it is also valid. We will therefore proceed by constructing the process as a limit of a sequence of processes. In this way positive semi-definiteness of the corresponding autocorrelation functions can be guaranteed, eliminating the difficulty of proving it directly.

The following theorem shows that the $\mathrm{AP}_{q,c}$ family corresponds to valid processes, and therefore proves for the first time that the familiar fractional noise class is not identical to that of second-order self-similar processes. Recall that $a|b$ means a divides b, i.e. $b = a * n$, $n \in \mathbb{Z}$.

Theorem 1 (New Second-Order Self-Similar processes) *Each member of the $\mathrm{AP}_{q,c}$ family is a second-order self-similar process.*

Proof
Define the periodic function $\bar{X}_m(t)$ as $\bar{X}_m(t) = 1$ if $m|t$, and 0 otherwise. A stationary process $X_m(t)$ can be defined as $X_m(t) := \bar{X}_m(t-k)$ where k is a random variable uniformly distributed on 0, 1, …, $m-1$. Now consider $Y_m(t) := X_m(t) - X_m(t-1)$. Since $\omega_m(n) := \omega_{Y_m}(n) = E[(Y_m(1) + Y_m(2) + \cdots + Y_m(n))^2] = E[(X_m(n) - X_m(0))^2]$, it is straightforward to show that $\phi_m(n) = 1$ except when $m|n$, where it vanishes. We now construct a new CTF via a convergent infinite sum of periodic CTFs of the above type:

$$\phi_{q,c}(n) := \frac{1-c}{c}\sum_{k=1}^{\infty} c^k \phi_{q^k}(n). \tag{10}$$

It is straightforward to verify that $\phi_{q,c}(n) = c^f$, where $n = aq^f$, $q \not| a$, $a \in \mathbb{Z}^+$, and f a non-negative integer. This however is nothing other than the CTF of $\mathrm{AP}_{q,c}$. It remains to show that the validity of covariance time functions is preserved under countable summation. Let F and G be independent processes with variance time functions ω_F and ω_G respectively. For simplicity, without loss of generality we assume $\mu = 0$. As $F + G$ is a process, its variance time function is valid, which as we now show equals $\omega_F + \omega_G$. According to equation (4), $\mathcal{V}_F^{(m)} = \omega_F(m)/m^2$ and $\mathcal{V}_G^{(m)} = \omega_G(m)/m^2$, and $\mathcal{V}_{F+G}^{(m)} = \frac{1}{m^2}E[((F(1) + \ldots + F(m)) + (G(1) + \ldots + G(m)))^2] = \mathcal{V}_F^{(m)} + \mathcal{V}_G^{(m)}$ which, using (4) again, means that $\omega_{F+G} = \omega_F + \omega_G$. For the infinite case, let $\gamma_k := \delta^2\{\omega_k\}$ and $\gamma := \delta^2\{\omega\}$. Since $\gamma_k(i)$ for any finite k is a finite linear combination of ω values, the pointwise convergence of ω_k implies that of γ_k. For any finite k, and any vector $\underline{\mathbf{a}}$ of finite length m, $R_{\underline{\mathbf{a}},k} := \sum_{j,i} \underline{\mathbf{a}}_j \gamma_k(j-i)\underline{\mathbf{a}}_i \geq 0$. As $R_{\underline{\mathbf{a}},k}$ is a finite linear combination of $\gamma_k(i)$ values the summation and limit operation commute, that is $\lim_{k\to\infty} R_{\underline{\mathbf{a}},k} = \sum_{j,i} \underline{\mathbf{a}}_j \gamma(j-i)\underline{\mathbf{a}}_i$. But as $R_{\underline{\mathbf{a}},k} \geq 0$ for all k its limit cannot be negative, proving the positive semi-definiteness of γ. The correlation time functions of definition 4 are therefore valid fixed points, as required. □

We conclude by examining more closely another member of the almost periodic family, that with $q = 3$ and $c = 0.5$. In the top left plot of figure 2, values up to $n = q^2$ are given in order to clearly show the entry of a new minimum value at $n = 9$, since $\phi(3^2) = \phi(3)^2$. In the second column many more lags are shown. At this scale one must look closely (consider $\phi(81)$ in the top middle plot) to see that the functions are not periodic. This is more readily visible in a log-log plot over a wide range of lags, as given in the rightmost graph. The dashed curve superimposed to the left of $\phi(n)$ corresponds to the power-law n^α, $\alpha = \log(c)/\log(q)$, on which the successive new minima of $\phi(n)$ fall when $n = q^j$ for some $j \in \mathbb{N}$.

Acknowledgements

The authors gratefully acknowledge the valuable comments of Prof. L. Győrfi.

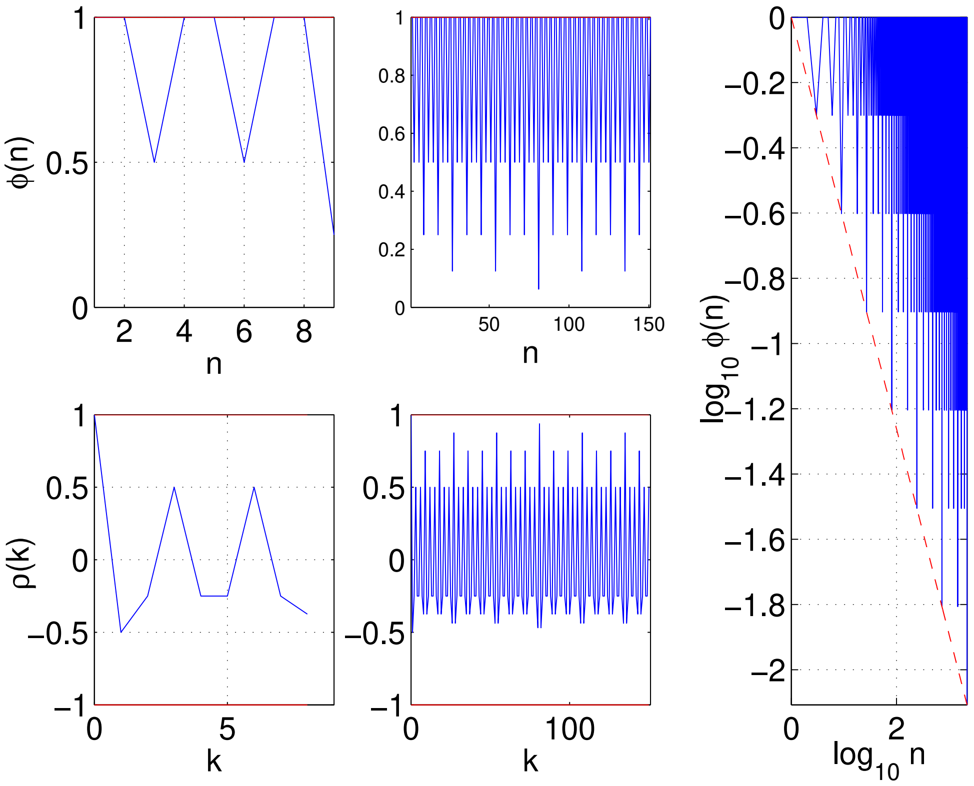

Figure 2: A closer look at $\mathrm{AP}_{q,c}$ with $q = 3$, $c = 0.5$. First column: the ϕ and ρ functions over the first few lags, Second column: over a wide range of lags each appears periodic, Third column: a log-log plot reveals the non-periodicity.

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