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A RANDOM MULTIFRACTAL MODEL WITH A GIVEN SPECTRUM

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□ *In this paper we present a novel method for the construction of a stochastic process whose multifractal spectrum has been prescribed. The main idea of the construction is to choose a suitable time-change reparametrizing the time of a fractional Brownian motion with an appropriately chosen Hurst parameter. Under certain conditions every given linear spectrum can be reproduced using this construction. Moreover, a wide range of Legendre spectra can be arbitrarily closely approximated. The properties of the model are also discussed and the possible use of the model is shown in a simulation study.*

Keywords Fractional Brownian motion; Multifractals; Multifractal spectrum; Time-change; Traffic modeling.

Mathematics Subject Classification Primary 28A80, 60G15, 60G17; Secondary 28A78, 60K30.

1. INTRODUCTION

The general concept of "fractality" encompasses a broad range of phenomena which share the common feature of displaying complex and irregular behavior on several different space and/or time scales[12]. The related mathematical theories (geometric and stochastic self-similarity, long-range dependence, extremal statistics, etc.) have found numerous applications in various scientific and engineering fields such as hydrodynamics, geophysics, biophysics, financial modelling, signal and image processing and, more recently, network traffic modelling[10,13,14].

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As far as the latter field of research – the one the authors are most familiar with – is concerned, fractal modelling techniques have been in use since the early 90's[10]. First, the presence of *long-range dependence* and *statistical self-similarity* was established in network measurement data and several traffic models, such as fractional Brownian motion (fBm)[16]. Fractionally integrated ARMA models, on/off models, were proposed to explain these properties. Further studies[1,7,19] showed that some types of data traffic (e.g., TCP traffic) have a more complex scaling behavior which cannot be described by long-range dependence and self-similarity especially on small time scales. It was shown that the aggregate network traffic is asymptotically self-similar over time scales of the order of magnitude of a few hundred milliseconds (a typical packet round-trip time in the network) and above, but below this time scale the variations in the data cannot be explained using a single parameter as it is the case for fBm. This time dependence of the scaling properties is called *multiscaling* in the engineering terminology. The mathematical objects able to model this kind of behavior are the *multifractals*.

The same transition from monofractals to multifractals can be observed within other fields like finance and geophysics[13,14]. To perform the statistical analysis of data with assumed multifractal properties, sophisticated estimation techniques were developed[2,18].

Several processes can be considered for multifractal modelling, *multiplicative cascades* being the simplest and most wide-spread ones[8,18]. Combining this process with the fBm model we can define a new class called the *fractional Brownian motion in multifractal time*[11]. This process has several nice properties, e.g., it is able to capture LRD and multifractal scaling independently. The *self-similar α-stable process*[20] is another option, but it has an irregular multifractal structure since its higher order moments are infinite. One of the simplest process from this class is the *linear fractional stable motion*[20].

In spite of the large number of studies using multifractal models, only a few papers address the issue of *generating* multifractal processes. As an example, Kant proposes a multifractal traffic generation method to obtain desired scaling and queueing properties based on a cascade construction[9]. Veitch et al. present a method for on-line generation of time-series with certain multifractal properties[21]. Our research aims at providing a generation method based on some characteristic descriptor of the multifractal process.

Multifractal processes can be characterized by a real function called the *multifractal spectrum* (for details see next section). This spectrum (not to be confused with the power spectrum in time series analysis) describes the "unevenness" of the complexities present in the process. For example, the fBm, whose scaling behavior can be described by one single parameter, has a multifractal spectrum consisting of only one point (more precisely its spectrum is non-zero for only one point). Our main question in this research is that if we are given a multifractal spectrum as an input, how can we construct a process which possesses this spectrum? The main idea of this construction is to find an appropriate time-change parametrization of a fBm with appropriately chosen Hurst parameter. The construction proposed in this paper is able to reproduce certain linear spectra. Moreover, a wide range of *Legendre* spectra can be arbitrary closely approximated.

The paper is structured as follows: In Section 2 we review the exact definitions of the multifractal spectrum. Starting with the approximation problem of a given Legendre spectrum, in Section 3 we present the analytical construction of the time-change (the main result of the paper) and present some further considerations related to the covariance structure of the process, stationarity, long-range dependence and possible generalizations. Section 4 provides some simulation results which validate our method. Finally, in the Appendix we add some further clarifying remarks concerning the model and provide a (highly technical) formal proof of the theorem.

2. THE BACKGROUND OF MULTIFRACTALS

As mentioned above, self-similar processes are the precursors of multifractals. In the stochastic setting, the simplest way of defining them is by prescribing the scaling of their moments[3]:

$$\mathbf{E}[|X(t)|^q] = c(q)t^{\tau(q)}, \quad \text{for all } t \in \mathcal{T},\ q \in \mathcal{Q}, \tag{1}$$

where $\mathcal{T}$ and $\mathcal{Q}$ are intervals on the real line, $0 \in \mathcal{T}$, $[0,1] \subseteq \mathcal{Q}$. $\tau(q)$ is called the *scaling function* and the prefactor $c(q)$ is independent of t.

Strictly speaking, fractality is related to the geometric shape of an object, so the first approach to multifractals is based on the study of the local erratic behavior of functions. For stochastic processes this can be done for the sample paths. Once the degree of the local irregularity for functions is defined, we will "measure" the set of points at which the function has the same degree of irregularity.

2.1. Hausdorff Spectrum

The Hausdorff β-measure of a set $G \subset \mathbb{R}$ is defined for $\beta > 0$ by

$$H^\beta(G) = \lim_{\varepsilon\to 0} \inf\left\{ \sum_{G\subset\bigcup_i I_i} (\lambda(I_i))^\beta : I_i \text{ closed interval and } \lambda(I_i) \leq \varepsilon \right\}, \tag{2}$$

where λ is the Lebesgue measure. It is known that there is a number β_0 such that $H^{\beta}(G) = +\infty$ if $\beta < \beta_0$ and $H^{\beta}(G) = 0$ if $\beta > \beta_0$ (the number $H^{\beta_0}(G)$ itself may vary between 0 and $+\infty$).

The Hausdorff dimension of the set $G \subset [0, 1]$ is given by

$$\dim_{\mathscr{H}} G = \inf\{\beta \geq 0 : H^{\beta}(G) = 0\} = \beta_0. \tag{3}$$

Let $Y(t)$, $t \in [0, 1]$ be a continuous function or a continuous sample path of a stochastic process. The definition of *local Hölder continuity exponent* of Y at time instant t is the following:

$$\mathscr{H}_Y(t) = \sup\{a : \exists\delta > 0, \exists c > 0, \forall s |s - t| < \delta, |Y(t) - Y(s)| \leq c|t - s|^{a}\} \tag{4}$$

To allow for more generality one can define the *lower* and the *upper grained Hölder-continuity exponents*:

$$\underline{h}_Y(t) = \liminf_{\varepsilon\to\infty} \frac{1}{\log_2(2\varepsilon)} \log_2 \sup_{|s-t|<\varepsilon} |Y(s) - Y(t)|$$

$$\overline{h}_Y(t) = \limsup_{\varepsilon\to\infty} \frac{1}{\log_2(2\varepsilon)} \log_2 \sup_{|s-t|<\varepsilon} |Y(s) - Y(t)|.$$

Let $\underline{E}_a^Y = \{t : \underline{h}_Y(t) = a\}$, $\overline{E}_a^Y = \{t : \overline{h}_Y(t) = a\}$, and $E_a^Y = \{t : \mathscr{H}_Y(t) = a\} = \underline{E}_a^Y \cap \overline{E}_a^Y$. The function Y is said to have multifractal structure if the E_a^Y sets are highly interwoven, each lying dense on the line[18]. Thus the *Hausdorff spectrum*, which is defined by

$$f_H^Y : [0, +\infty) \ni a \mapsto \dim_{\mathscr{H}} E_a^Y \in [0, 1], \tag{5}$$

describes the connection between the local variability and the measure of the set of all points having the same given local variability. The Hausdorff spectrum is also termed the *multifractal spectrum* in the "classical" fractal literature.

2.2. Grained Spectrum

An alternative approach to the above description of multifractals consists in replacing the Hausdorff dimension by the familiar *box dimension*[6]: we count the coarse exponents at resolution n sufficiently close to the given Hölder exponent and take appropriate limits of the rescaled logarithmic number thus obtained. This method is closely related to the probabilistic law of large numbers and large deviation theory, yielding the *coarse grained* or *large deviation spectrum.*

Consider again a function $Y(t)$ defined on $I = [0, 1]$. For any n this interval can be decomposed as the disjoint union of 2^n dyadic subintervals $I_n^k = [k2^{-n}, (k+1)2^{-n})$, where $k = 0, 1, \ldots, 2^n - 1$. The *coarse Hölder exponents* at resolution n are defined as the rescaled logarithmic increments of the process on I_n^k, i.e., $\alpha_n^k = -\frac{1}{n}\log_2 \left| Y\big((k+1)2^{-n}\big) - Y(k2^{-n}\big)\right|$.

The large deviation spectrum measures, loosely speaking, how fast the probability of observing a coarse Hölder exponent different from the expected value tends to zero as the resolution tends to infinity[11].

Let $N_n^\varepsilon(\alpha)$ be of the form

$$N_n^\varepsilon(\alpha) = \#\big\{k = 0, 1, \ldots, (2^n - 1) : \alpha_n^k \in [\alpha - \varepsilon, \alpha + \varepsilon]\big\}, \tag{6}$$

where $\#A$ denotes the cardinality of the finite set A and $\varepsilon > 0$ is the scale of measurement. Then the large deviation spectrum, denoted by f_G^Y, is defined by

$$f_G^Y(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N_n^\varepsilon(\alpha)}{n}. \tag{7}$$

The large deviation spectrum describes the distribution of the local singularities, since the number of dyadic intervals of size 2^{-n} with coarse Hölder exponent *close to* α varies roughly on a logarithmic scale as $2^{nf_G^Y(\alpha)}$ for large n. With the usual informal notation

$$\mathrm{P}_n\left[\alpha_n^k \approx \alpha\right] \asymp 2^{-n(1-f_G^Y(\alpha))}, \tag{8}$$

where the probability is related to a random choice of k uniformly in $\{0, 1, \ldots, (2^n - 1)\}$, i.e., P_n is the uniform distribution on the set of all dyadic intervals I_n^k of size 2^{-n}.

2.3. Legendre Spectrum

The Legendre spectrum provides a robust estimation of the large deviation spectrum when the data satisfies some necessary conditions[11]. Let $Z_n^k = |Y((k+1)2^{-n}) - Y(k2^{-n})|$, $k = 0, 1, \ldots, 2^n - 1$ denote the discrete increment process of Y. Define the *partition sum* $S_n(q)$ with $q \in \mathbb{R}$:

$$S_n(q) = \sum_{k=0}^{2^n-1} |Y((k+1)2^{-n}) - Y(k2^{-n})|^q = \sum_{k=0}^{2^n-1} (Z_n^k)^q. \tag{9}$$

Then the so-called *scaling function* can be given by

$$\tau(q) = \liminf_{n\to\infty} \frac{\log S_n(q)}{-n\log 2}, \tag{10}$$

and thus the *Legendre spectrum* of Y is defined by the following

$$f_L(\alpha) = \tau^*(\alpha) = \inf_{q\in\mathbb{R}}(\alpha q - \tau(q)). \tag{11}$$

Consider the moment-generating function of the random variable $X_n = \log Z_n^K$ where K is uniformly distributed on $\{0, 1, \ldots, 2^n - 1\}$

$$c_n(q) = -\frac{1}{n}\log \mathbf{E}_n[e^{qX_n}] = -\frac{1}{n}\log(2^{-n}S_n(q)). \tag{12}$$

The Gärtner–Ellis theorem[5] shows that if $\lim_{n\to\infty} c_n(q)$ exists (in which case $c_n(q) = \tau(q) + 1$) and differentiable, then the following relation holds

$$f_L(\alpha) = f_G(\alpha), \tag{13}$$

The Legendre spectrum provides a method for detection and delineation of multifractal properties. Moreover, it is favorable since its computation is much easier than the direct computation of the large deviation spectrum which requires the evaluation of local quantities and of a double limit[11]. This description of multifractality is used in our study.

3. RANDOM MULTIFRACTALS WITH GIVEN SPECTRA

3.1. The Model Construction

Fractional Brownian motion (fBm) is the best known fractal process. The local Hölder continuity exponent of the fBm is the same and equal to the Hurst parameter for all points all along its trajectory with probability 1. In other words, fBm is a monofractal, i.e., its different multifractal spectra (f_H, f_G, f_L) are non-zero for only one point. Our proposed multifractal model is a suitably time-changed fBm process. We note that the idea of time-changing a multifractal process was first introduced by Mandelbrot and Taylor in[15].

Our aim is to modify the trajectory of the fBm at one point using a very simple function. More precisely, let $\phi_t(x) = \text{sign}(x-t)|x-t|^\alpha + C_0$ where C_0 is a real number ensuring that ϕ_t is positive on the positive real line. In the sequel we call ϕ_t the *elementary time-change function around t of order α*.

Denote by $B(t)$ a fractional Brownian motion with Hurst parameter H (the dependence on this parameter is suppressed in the notation). Consider the time-changed fBm $B(\phi_t(x))$, $x \geq 0$. Due to the time-change the Hölder continuity exponent of all the trajectories is unchanged except at t where the exponent is equal to αH. Thus, depending on α, we can accelerate or slow down "the velocity of the variability of the fBm" at t. However, we cannot affect the spectrum because to do so the Hölder continuity exponents should be changed at uncountably many points.

The first step consists in the construction of a process with a given *linear* spectrum.

The next result proves to be one of the most useful tools in this construction. It implies that for a time-changed fBm we only have to find a continuous increasing function as a time-change such that the given spectrum is the scaled-down version of the spectrum of this time-change.

Theorem 3.1.1 (Ref.[18]). *Let B be a fBm with Hurst-parameter H and $\mathcal{M}$ be an almost surely continuous random time-change independent of B, and set $\mathcal{B}(t) := B(\mathcal{M}(t))$. For almost every path the following fact is true for any $a > 0$*

$$f_G^{\mathcal{B}}(a) = f_G^{\mathcal{M}}(a/H). \tag{14}$$

Let α, δ be two positive real numbers such that $\delta < \alpha$ and $0 < \delta < 1$. An important result of this paper is the construction of a time-change function $F_{\alpha,\delta}$, called the *basic time-change with parameter* (α, δ), whose grained spectrum and the Legendre spectrum consists of a line connecting the points $(1, 1)$ and (α, δ). By Theorem 3.1.1 the spectrum of $B(F_{\alpha,\delta}(x))$, $0 \le x \le 1$, connects $(H, 1)$ and $(\alpha H, \delta)$. In our construction $F_{\alpha,\delta}$ is an integral of the $(\alpha - 1)$th power of a core function Φ_δ depending only on $\delta : F_{\alpha,\delta}(x) = \int_0^x (\Phi_\delta(y))^{\alpha-1} dy$, $x \in [0, 1]$. The precise construction is presented in the next subsection.

Next the method to approximate an arbitrary given Legendre spectrum f_L is presented. (This type of spectrum is considered because it can be easily calculated in practice.) Suppose that f_L reaches its maximum at exactly one point H, $f_L(H) = 1$, and $f_L(\alpha) \le \alpha/H$ for any α, i.e., the whole spectrum lies under the straight line which goes through the origin and has slope $1/H$ (see Figure 1). Let (A, Δ) be a set of some arbitrarily chosen points on

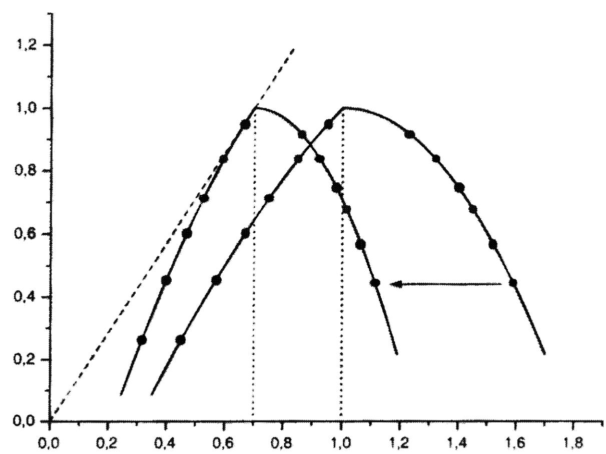

FIGURE 1 The spectrum of the time-change function and the time-changed fBm ($H = 0.7$).

the graph of $f_L : (A, \Delta) = \{(\alpha_i, \delta_i)\}_{i=1}^n$. We will construct a process $\mathscr{B} := \mathscr{B}_{A,\Delta}$ whose Legendre spectrum and grained spectrum are the ones given at the points $\alpha_i : f_G^{\mathscr{B}}(\alpha_i) = \delta_i$ $(1 \leq i \leq n)$ and the spectrum equals 1 at the point H and $f_G^{\mathscr{B}}(x) < f_L(x)$ for the remaining points $x \neq \alpha_i$.

Since the Legendre spectrum $f_L^{\mathscr{B}}$ is the concave hull of the grained spectrum $f_G^{\mathscr{B}}$ (see, e.g., Ref.[18]) any given Legendre spectrum f_L can be arbitrary closely approximated by $f_L^{\mathscr{B}}$, see details in Figure 1.

Instead of constructing $F_{A,\Delta}$ we first construct $F_{\alpha,\delta}$ on $[0, 1]$ for any pair of (α, δ). After that joining the individual functions $F_{\alpha,\delta}$, $(\alpha, \delta) \in (A, \Delta)$ while keeping the continuity will result $F_{A,\Delta}$. Clearly, this provides the spectrum depicted above.

The detailed procedure of this construction is given in Subsection 3.3. One can find more sophisticated but only heuristical methods for getting suitable time-change in Subsection 3.4.

3.2. The Basic Time-Change Function $F_{\alpha,\delta}$

3.2.1. *The Irregularities of $F_{\alpha,\delta}$ and the Core Function H_δ*

We present a Cantor-type fractal construction taken from[6]. Given a constant $0 < C < 1/2$, set out from the interval $[0, 1]$ and remove the interval $(C, 1 - C)$ from it. In the next step, remove the middle intervals from $[0, C]$ and $[1 - C, 1]$ proportionally as in the first step, more precisely $(C^2, C(1 - C))$ and $(1 - C + C^2, 1 - C + C(1 - C))$. Going on indefinitely with this procedure we get the set $\mathscr{E}$. The Hausdorff dimension of this set is δ, the solution of the equation $2C^\delta = 1$.

For a given δ compute C by means of the previous equation. To define the set $\mathscr{E}$ let us define the following sequences of embedded sets containing the endpoints of the removed subintervals, namely

$$\mathscr{E}_0 = \{0, 1\}, \quad \mathscr{E}_1 = \{0, C, 1 - C, 1\}, \ldots, \mathscr{E}_{i+1} = C\mathscr{E}_i \cup (1 - C + C\mathscr{E}_i), \ldots$$

and denote $\mathscr{E}^* = \bigcup_{i=0}^{\infty} \mathscr{E}_i$. Furthermore, we distinguish between two types of points in $\mathscr{E}^*$. In the sequel, let right(left)-hand side point $\mathscr{R}(\mathscr{L})$ mean that any points of $\mathscr{R}(\mathscr{L})$ is a right(left)-hand side endpoint of a closed interval appearing in some step during the construction of $\mathscr{E}$. $\mathscr{P}_i := \mathscr{E}_i \backslash \mathscr{E}_{i-1}$ denotes the set of the new endpoints in $\mathscr{E}_i$ coming up in the ith step. We also separate the right-hand side and the left-hand side endpoints in $\mathscr{P}_i : \mathscr{R}_i = \mathscr{R} \cap \mathscr{P}_i$ and $\mathscr{L}_i = \mathscr{L} \cap \mathscr{P}_i$.

3.2.2. *The Heuristical Construction of the Function $F_{\alpha,\delta}$*

First of all, remember that the function

$$\mathrm{sign}(x - t)|x - t|^\alpha + C_0$$

has local Hölder-continuity exponent α at point t. It proves to be the most important fact for understanding the construction and the proof.

Now, suppose that α and δ, and C are given. Let $K_C = \min\{C, 1-2C\}$ and $r_i = C^i K_C$. For $t \in \mathcal{P}_i$ let $I(t) = [t - r_i, t + r_i]$ be the so-called modificational interval around t. (For a better understanding of the choice of these quantities read the Remarks 1 and 2 in Section 6.) Define the modifying function $g_t(x)$ around t, $t \in \mathcal{P}_i$, such that

$$g_t(x) = \begin{cases} 0, & \text{if } x \leq t - r_i, \\ f^{(i-1)}(x) - \operatorname{sign}(x-t)|x-t|^{\alpha} - f^{(i-1)}(t-r_i) - r_i^{\alpha}, & \text{if } t \in I(t), \\ f^{(i-1)}(t+r_i) - f^{(i-1)}(t-r_i) - 2r_i^{\alpha}, & \text{if } x \geq t + r_i. \end{cases} \tag{15}$$

Using these g_t's define the following sequence of functions on $[0,1]$:

$$\begin{aligned} f^{(0)}(x) &= x \\ f^{(i)}(x) &= f^{(i-1)}(x) - \sum_{t \in \mathcal{P}_i} g_t(x) \end{aligned} \tag{16}$$

The desired $F_{\alpha,\delta}$ of Theorem 3.2.3.1 exists and it is built as the limit of these $f^{(i)}$s. For the proof see the remark after Theorem 3.2.3.1.

3.2.3. *The Integral Construction of the Function $F_{\alpha,\delta}$*

To define the core function Φ_δ let us introduce the following sequence of functions $\{\Phi_\delta^{(i)}\}_{i=1}^{\infty}$ on $[0,1]$ by the next recursion (see Figure 2 for example):

$$\Phi_\delta^{(0)}(x) = 1, \quad x \in \mathbb{R}$$

$$\Phi_\delta^{(i)}(x) = \begin{cases} \Phi_\delta^{(i-1)}(x), & \text{if } x \in [0,1] \backslash \Big(\bigcup\limits_{t \in \mathcal{P}_i} I(t) \Big) \\ |x-t|, & \text{if } x \in I(t) \text{ for some } t \in \mathcal{P}_i \end{cases}, \quad i \geq 1$$

The limit of Φ_δ functions exists since $\{\Phi_\delta^{(i)}\}_{i=1}^{\infty}$ is a decreasing sequence of non-negative functions. Naturally, we have the following identity:

$$F_{\alpha,\delta}(x) = \lim_{i\to\infty} f^{(i)}(x) = \int_0^x (\Phi_\delta(y))^{\alpha-1} dy. \tag{17}$$

The advantage of the usage of such an integral representation is that we can separate the effect of the component α (the local Hölder-continuity exponent) and the effect of the component δ (the Hausdorff measure).

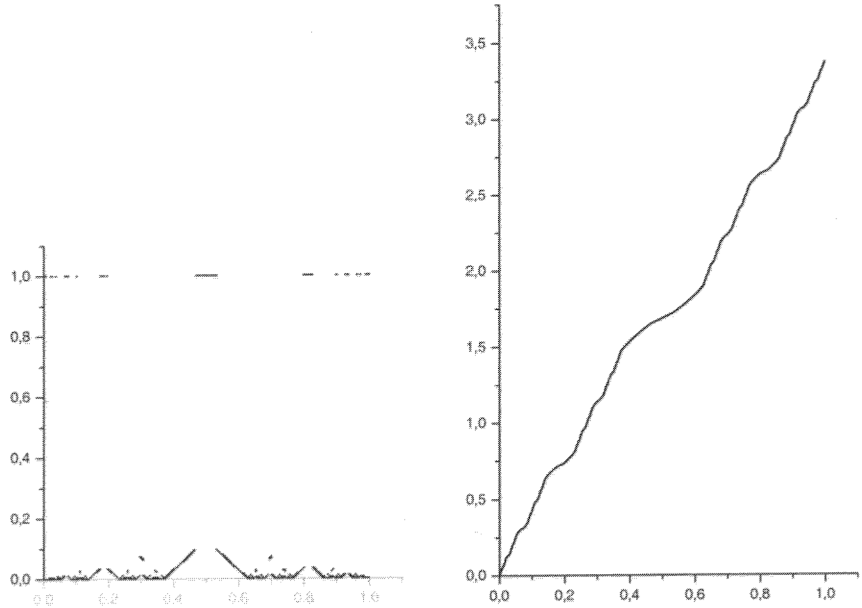

FIGURE 2 An example: $\Phi_{0.7}$ and $F_{0.7,0.9}$.

Our main result is stated by the following theorem:

Theorem 3.2.3.1. *For any given α and δ, $\delta < \alpha$, $F = F_{\alpha,\delta}$ makes sense and the following facts are true for F:*

$$\text{If } 1 < \alpha \quad \text{then } f_G^F(\beta) = \begin{cases} -\infty, & \text{if } \beta < 1, \\ 1 - (\beta - 1)\dfrac{1-\delta}{\alpha - 1}, & \text{if } 1 \le \beta \le \alpha, \\ -\infty, & \text{if } \alpha < \beta. \end{cases} \tag{18}$$

$$\text{If } \delta < \alpha < 1 \quad \text{then } f_G^F(\beta) = \begin{cases} -\infty, & \text{if } \beta < \alpha, \\ 1 - (\beta - 1)\dfrac{1-\delta}{\alpha - 1}, & \text{if } \alpha \le \beta \le 1, \\ -\infty, & \text{if } 1 < \beta. \end{cases} \tag{19}$$

The proof can be found in Section 6.2.

An illustration of the spectrum and the scaling function of the $F_{\alpha,\delta}(\cdot)$ function is shown in Figure 3. We distinguish between two main cases: $\alpha < 1$ and $\alpha > 1$. The related spectrum and the corresponding scaling function can be seen in Figure 3(a) for the case $\alpha < 1$ and in Figure 3(b) for the case $\alpha > 1$.

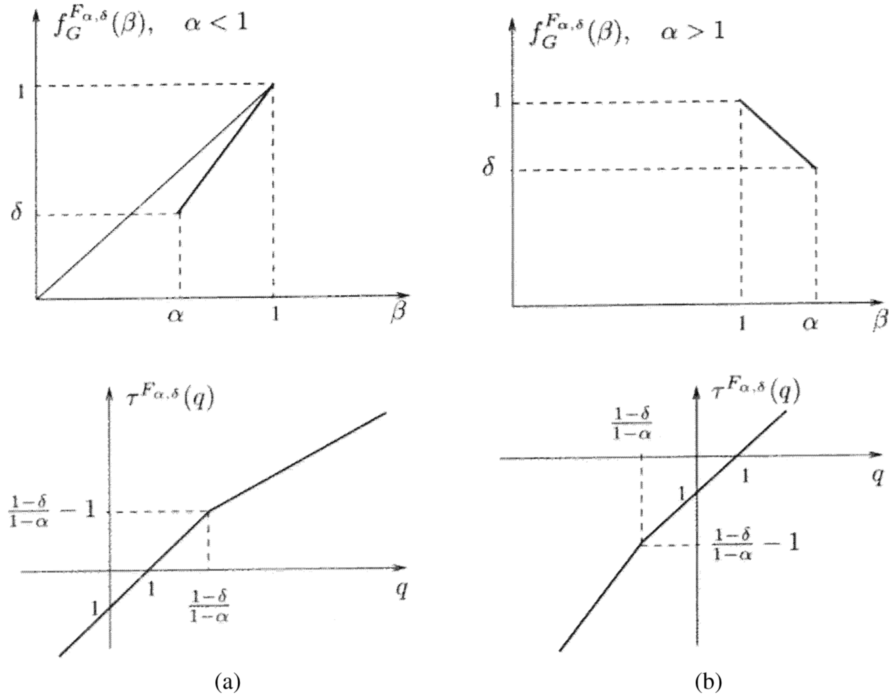

FIGURE 3 The grained spectra and the scaling functions of $F_{\alpha,\delta}$ with different parameters.

The scaling function of the time-changed fBm of Hurst parameter H with the time-change function $F_{\alpha,\delta}(\cdot)$ is given in Figure 4.

3.3. Approximation of a Given Spectrum

Recall from Section 2 that the scaling function $\tau(q)$ is defined by $\mathbf{E}[|X(t)|^q] = c(q)t^{\tau(q)}$. Assume that the scaling function $\tau(q)$ is given. Our

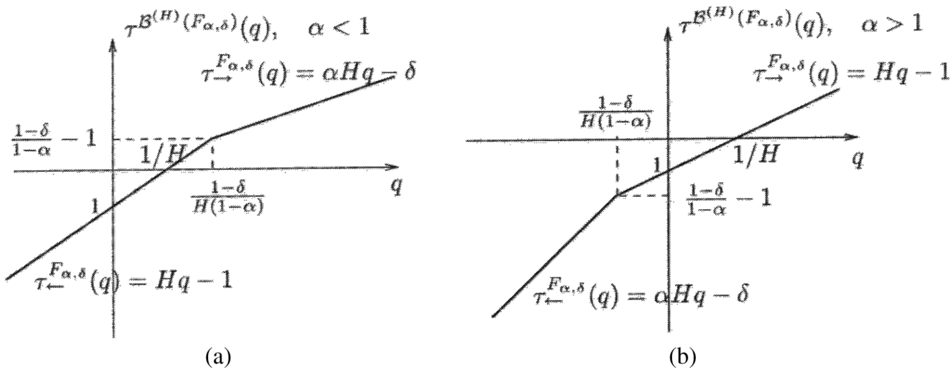

FIGURE 4 Scaling function of the time-changed fBm with time-change function $F_{\alpha,\delta}$.

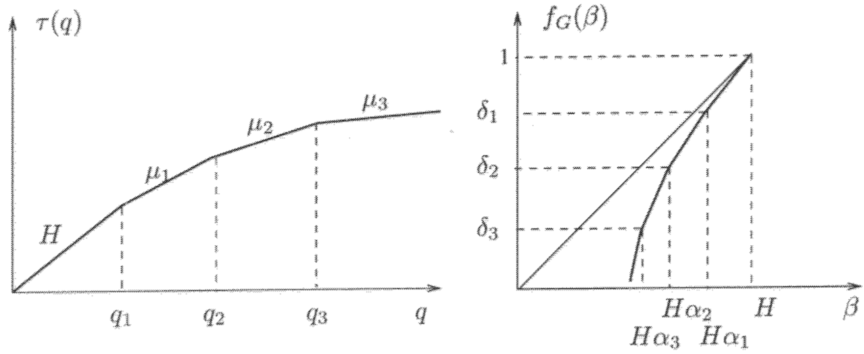

FIGURE 5 Approximation of a given spectrum.

proposed method approximates $\tau(q)$ by a piecewise linear function, which is characterized by the set

$$\{H, (q_1, \mu_1), \ldots, (q_k, \mu_k)\}, \tag{20}$$

where H is the initial slope of τ, q_i's are the breakpoints, and μ_i's are the slopes of τ on the corresponding intervals $[q_i, q_{i+1}]$'s (see Fig. 5). With the set $\{H, (q_1, \mu_1), \ldots, (q_k, \mu_k)\}$ our task is to determine the sequence

$$\{(\alpha_1, \delta_1), \ldots, (\alpha_k, \delta_k)\} \tag{21}$$

from which one can build F_{α_i,δ_i}-s and the time-change function $F_{A,\Delta}$ as aforementioned. Let $\mathscr{B}^{(H)}$ be a fBm with parameter H given above. The Legendre transform of the scaling function of the process $\mathscr{B}^H(F_{A,\Delta})$ will give the desired approximation of τ.

Denote by $\tau^{\mathscr{B}^{(H)}(F_{A,\Delta})}$ and $\tau^{F_{A,\Delta}}$ the scaling function of $\mathscr{B}^{(H)}(F_{A,\Delta})$ and $F_{A,\Delta}$, respectively. By the relation

$$\tau^{F_{A,\Delta}}(q) = \tau^{\mathscr{B}^{(H)}(F_{A,\Delta})}(q/H) = \tau(q/H)$$

and by the definition of Legendre transformation we can evaluate the breakpoints of the Legendre spectrum of $F_{A,\Delta}$ (cf. Eq. (21)):

$$\{(\mu_1/H, \delta_0 - q_1(\mu_0 - \mu_1)), \ldots, (\mu_k/H, \delta_{i-1} - q_i(\mu_{i-1} - \mu_i))\}, \quad \delta_0 = 1.$$

Recall from convex analysis[5] that for strictly concave functions the double application of Legendre transform returns the original function, i.e.,

$$f(a) = \inf_q(aq - \tau(q))$$

Elementary calculations show that the spectra corresponding to Eq. (20) and the spectrum of $\mathscr{B}^{(H)}(F_{A,\Delta})$ with breakpoints $(A,\Delta) = \{(\alpha_1,\delta_1),\dots,(\alpha_k,\delta_k)\}$ are the following

$$f_L(a) = \begin{cases} -\infty & \text{if } a < \mu_k \\ q_k a - \tau(q_k) & \text{if } \mu_k \le a < \mu_{k-1} \\ \dots & \dots \\ q_2 a - \tau(q_2) & \text{if } \mu_2 \le a < \mu_1 \\ q_1 a - \tau(q_1) & \text{if } \mu_1 \le a < H \\ 1 & \text{if } H = a \\ -\infty & \text{if } H < a \end{cases}$$

$$f_L^{\mathscr{B}^{(H)}(F_{A,\Delta})}(a) = \begin{cases} -\infty & \text{if } a < H\alpha_k \\ \dfrac{\delta_{k-1}-\delta_k}{H\alpha_{k-1}-H\alpha_k}a+\delta_k & \text{if } H\alpha_k \le a < H\alpha_{k-1} \\ \dots & \dots \\ \dfrac{\delta_1-\delta_2}{H\alpha_1-H\alpha_2}a+\delta_2 & \text{if } H\alpha_2 \le a < H\alpha_1 \\ \dfrac{1-\delta_1}{H-H\alpha_1}a+\delta_1 & \text{if } H\alpha_1 \le a < H \\ 1 & \text{if } H = a \\ -\infty & \text{if } H < a \end{cases}$$

Solving the equation $f_L(a) = f_L^{\mathscr{B}^{(H)}(F_{A,\Delta})}(a)$ we get a sequence of equations:

$$\begin{aligned} \alpha_i &= \mu_i/H \\ q_i &= \frac{\delta_{i-1}-\delta_i}{\alpha_{i-1}-\alpha_i}\frac{1}{H} \quad \text{for } 1 \le i \le k, \quad \delta_0,\ \alpha_0 := 1, \end{aligned} \tag{22}$$

from which

$$\delta_i = \delta_{i-1} - q_i(\mu_{i-1}-\mu_i) \quad \text{for } 1 \le i \le k.$$

One can find more advanced methods to get suitable time-changes. In this paper we will not present any exact results on general joining. The next proposition shows a difficulty about this sort of consideration.

Proposition 3.3.1. *Let $X(t)$ and $Y(t)$ be two increasing continuous functions on the real line starting from zero. The following facts hold on the local Hölder-*

continuity exponents:

$$\mathcal{H}_{X+Y}(t) = \min\{\mathcal{H}_X(t), \mathcal{H}_Y(t)\}, \quad \mathcal{H}_{XY}(t) = \min\{\mathcal{H}_X(t), \mathcal{H}_Y(t)\}$$

for $t > 0$.

It means that any additive and multiplicative method combining $F_{\alpha,\delta}$–s with different parameters may ignore the slow-downs in the resulted time-change functions. Therefore, working with Φ_δ's (in Subsection 3.2.3) could be more promising. For example, one can choose two pairs of parameters and build $\mathcal{M}(x) = \int_0^x \frac{\Phi_{\delta_1}^{\alpha_1-1}\Phi_{\delta_2}^{\alpha_2-1}}{\max\left\{\Phi_{\delta_1}^{\alpha_1-1}, \Phi_{\delta_2}^{\alpha_2-1}\right\}}(y)\,dy$ as the time-change function on $[0, 1]$.

3.4. Properties of the Proposed Multifractal Model

1. *The effect of* $\operatorname{sign}(x-t)|x-t|^\alpha + C_0$ *like functions* can be depicted as follows. If $\alpha > 1$ or $\alpha < 1$ $\operatorname{sign}(x-t)|x-t|^\alpha$ slows down or accelerates the process at the point t and so increases or decreases respectively the local Hölder continuity exponent.
2. *Properties of the time-changed fBm:*

   (2a) Easy calculation shows the correlation structure of $\mathcal{B}$ is the following:

   $$r(t, s) := \operatorname{Cov}[\mathcal{B}(t), \mathcal{B}(s)] = K_H\left[F(t)^{2H} + F(s)^{2H} - |F(t) - F(s)|^{2H}\right]$$

   for an appropriate constant K_H.

   (2b) $\mathcal{B} = B(F)$ is a Gaussian process which can be seen via its finite dimensional characteristic function (Ref.[20] Section 7.2):

   $$E\left[\exp\left(i\sum_{k=1}^{n} x_k\mathcal{B}(t_k)\right)\right] = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} r(t_k, t_l)x_k x_l\right)$$

   (2c) To get a process on the whole positive real line one can continue $F_{\alpha,\delta}$ beyond 1 by joining other only just dilated $F_{\alpha,\delta}$-s such that the joined function $F_{\alpha,\delta}^{[0,\infty)}$ remains continuous. It is easy to evaluate that the process $\mathcal{B}^\infty = B(F_{\alpha,\delta}^{[0,\infty)})$ inherits the long-range dependent property of B, i.e., $\mathcal{B}^\infty$ is *long-range dependent* if $1/2 < H < 1$.

3. *Extra randomization:* If multifractal process with non-Gaussian marginal is desired there are two straightforward possibilities. First, one can apply an extra random time-change $\mathcal{R} : [0, 1] \to [0, 1]$ to reparameterize $\mathcal{B}$ as $B(\mathcal{M}(\mathcal{R}_t))$ and $B(\mathcal{R}_t\mathcal{M}_t)$. Second, we can take the support of

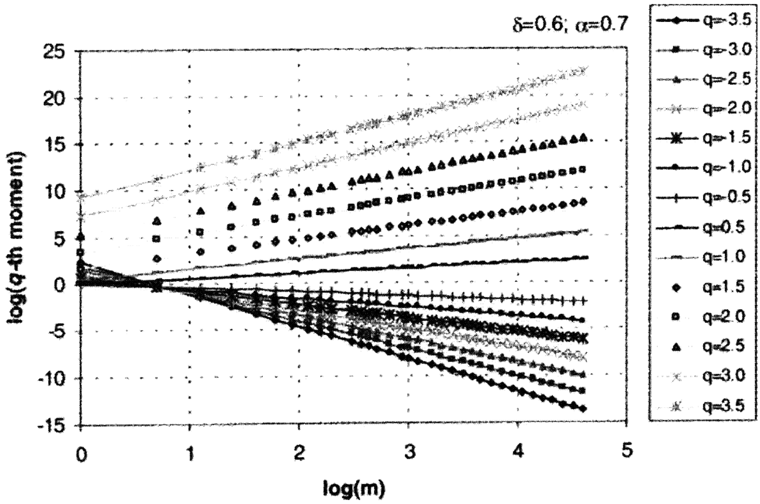

FIGURE 6 Estimation of scaling function using the moment method.

the irregularities of F randomly (random Cantor set), see (Ref.[6], Chapter 15). Following this trace the properly modified version of Theorem 3.2.3.1 should be proved, which seems to be rather sophisticated. However, in the first case the spectra of the new processes are the same as those of the original ones.

4. SIMULATION RESULTS

Simulation validation of our new multifractal model is presented in this section. To demonstrate the effectiveness of the method we generate a time-change function and simulate the time-changed fBm and then estimate the multifractal spectrum.

4.1. Simulation of the Basic Time-Change Function

As we have discussed, there are two typical cases of the time-change function $F_{\alpha,\delta}$ when $\alpha < 1$ and $\alpha > 1$. We simulated 2^{18} samples for both cases. The parameter pair (α, δ) was set as $(0.7, 0.6)$ and $(1.4, 0.8)$.

The scaling functions of the generated multifractal time-change functions were estimated using the simple moment method (see Ref.[4] for more details). The result of this method for the function $F_{0.7,0.6}$ is shown in Figure 6. The lines for the moments of different orders q indicate the existence of scaling properties in the process. The scaling function is estimated from the slopes of these lines.

Estimation of the scaling functions of the mentioned two cases is presented in Figure 7. Both figures consist of two line segments connected

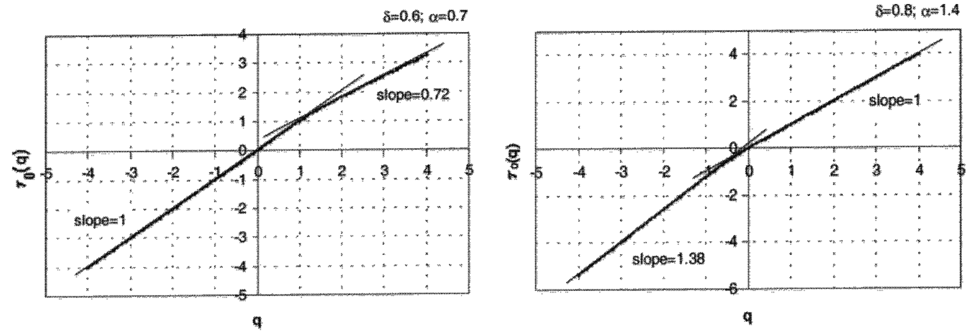

FIGURE 7 Scaling function of simulated $F_{\alpha,\delta}$ with $(\alpha,\delta) = (0.7, 0.6)$ (left) and $(1.4, 0.8)$ (right).

by a breakpoint. In the case of $F_{0.7,0.6}$ the lines have slopes 1 and 0.72. These values are 1.38 and 1 in the case of $F_{1.4,0.8}$. Comparing the results with the theoretical calculation presented in Figure 3 we can see that these results are almost exactly what we expected.

4.2. Generation of Multifractals with the Given Spectrum

Now, turn back the problem from the original point of view: suppose that we are given a process with multiscaling properties which is characterized by its scaling function. The process can be the aggregate traffic of high speed backbone network considered at small timescales; see, e.g., Ref.[7] for details. Our aim is to create a multifractal model with approximated scaling function as discussed.

Suppose that the given scaling function is approximated by a piecewise linear function with slopes 0.7 for moment order $0 \leq q < 2$, 0.56 for $2 \leq q < 3$, and 0.5 for $3 \leq q$ (see Figure 5). This means that the Hurst parameter of the fBm we should use in the model is 0.7. Furthermore, using the results of Eq. (22) the values of parameters of two time-change functions, which are responsible for the slopes 0.56 and 0.5, respectively, are $(\alpha_1, \delta_1) = (0.8, 0.72)$ and $(\alpha_2, \delta_2) = (0.72, 0.55)$.

The reparameterization procedure is done as follows: we use the fast Fourier transform-based routine provided by Paxson[17] to generate the incremental process of the fBm. Thus we have the function $B(t)$ for integer values of t. We then normalize and round the samples of time-change functions to make integer samples. Then the reparameterization of $B(t)$ by time-change functions is straightforward.

The scaling function of the data series generated by our multifractal model is given in Figure 8. Two breakpoints at moment order $q = 2$ and 3 can be observed as we expected. The slopes of the component lines of the plot are 0.7, 0.59, and 0.53 for the moment order interval $[0, 2)$, $[2, 3)$, and $[3, +\infty)$, respectively. This result, disregarding the small deviations

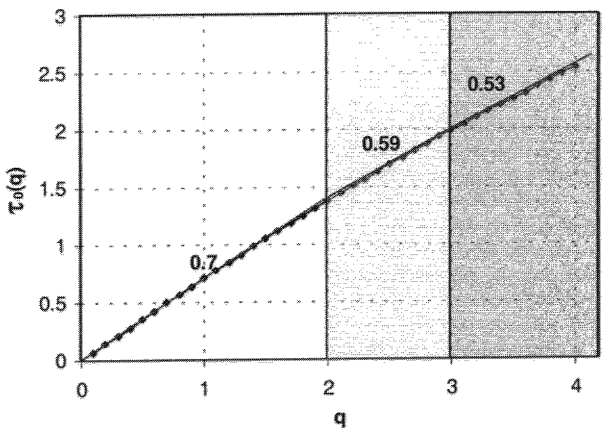

FIGURE 8 The scaling function of a simulation example of the model fitting a given scaling function.

for the two later slopes, verifies that the multifractal model matches the given multifractal characteristics. The reason for the slight deviations from the expected slopes (0.56 and 0.5) may be due to the method used for fBm reparameterization (explained above), which is definitely not an exact procedure. An exact solution is under development.

5. CONCLUSION

We presented a new multifractal model which is able to approximately reproduce a given multifractal spectrum. The model consists of an appropriately time-changed version of a fractional Brownian motion. We gave a detailed description for the construction of time-change function used in the model. The construction is verified in a simulation study and it is shown that the model can be easily applied in practice.

The proposed multifractal model can be useful in simulation and analytical investigation of wide-area network traffic having long-range dependent and multiscaling properties.

6. APPENDIX

6.1. Remarks

In the following section we add some clarifying remarks concerning the construction of the multifractal process.

6.1.1. *The Definiteness of $F_{\alpha,\delta}$ and the Meaning of the Condition $\delta < \alpha$*

We will show that the function $F_{\alpha,\delta}$ is finite for all α, δ if and only if $\delta < \alpha$. Since $F_{\alpha,\delta}$ is an increasing function on $[0, 1]$ if the previous statement is true for $F_{\alpha,\delta}(1)$, i.e., $F_{\alpha,\delta}(1) < \infty$ if $\delta < \alpha$ and $F_{\alpha,\delta}(1) = \infty$ if $\delta \geq \alpha$.

If $1 < \alpha$, the finiteness is straightforward because $f^{(i)}$ is a decreasing sequence of functions.

If $\delta < \alpha < 1$, the increment of $f^{(i)}$ in each step can be appropriately estimated from above. Recall from Eq. (16) the increment is $-\sum_{t\in\mathscr{P}_i} g_t(1)$ in the ith step which is estimated from above by $-\sum_{t\in\mathscr{P}_i} g_t(1) < 2^i 2r_i^\alpha = 2K_C^\alpha(2C^\alpha)^i$. $F_{\alpha,\delta}(1) < 1 + 2K_C^\alpha \sum_{i=1}^\infty (2C^\alpha)^i$ is finite because $2C^\alpha = 2^{1-\alpha/\delta} < 1$ $(\delta < \alpha)$.

In the reversed case, if $\delta \geq \alpha$ then $F_{\alpha,\delta}(1) = \infty$. For all $t \in \mathscr{P}_i$ we have

$$\begin{aligned} f^{(i-1)}(t + r_i) - f^{(i-1)}(t - r_i) &\leq (C^i + K_C C^i)^\alpha - (C^i - K_C C^i)^\alpha \\ &= [(1 + K_C)^\alpha - (1 - K_C)^\alpha] C^{i\alpha}. \end{aligned}$$

Hence, $g_t(1) \leq \left[\left((1+K_C)^\alpha - (1-K_C)^\alpha\right) - 2K_C^\alpha\right] C^{i\alpha} = -\vartheta_C C^{i\alpha}$ for some $\vartheta_C > 0$. Thus, $-\sum_{t\in\mathscr{P}_i} g_t(1) > \vartheta_C 2^i C^{i\alpha}$. In the case $\delta \geq \alpha$ we have the estimation $F_{\alpha,\delta}(1) \geq \vartheta_C \sum_{i=1}^\infty (2C^\alpha)^i = \infty$, that is, $F_{\alpha,\delta}$ does not make sense in this case.

After all, note that the necessity of the criteria $\delta < \alpha$ comes from the relation $f_G^{F_{\alpha,\delta}}(\beta) \leq \beta$, which is what we expect for increasing functions, c.f.[18].

6.1.2. *The Choice of the Approximating Basic Time-Change Function $f^{(i)}$*

First, we ensure the local Hölder continuity exponent α at the points of $\mathscr{P}_i$ by inserting the functions $\operatorname{sign}(x-t)|x-t|^\alpha$ on the intervals $I(t) = [t - r_i, t + r_i]$. Formally, it is achieved by subtracting the functions $\sum_{t\in\mathscr{P}_i} g_t$ from $f^{(i-1)}$. This implies that the Hölder continuity exponent of $F_{\alpha,\delta}$ is α at the points of $\mathscr{E}^* = \bigcup \mathscr{P}_i$. As Theorem 3.2.3.1 implies, one can extend this Hölder continuity exponent from $\mathscr{E}^*$ to $\mathscr{E}$. This is interesting since $\mathscr{E}^*$ is countable while $\mathscr{E}$ is not.

Next, there are two criteria that restrict the values of K_C. On one hand, we have to avoid the modification of the Hölder continuity exponent in some neighborhood of the points of $\mathscr{E}_{i-1}$ when we are accomplishing the ith refinement. On the other hand, the refining intervals at the same level must be disjoint so that the definition of $f^{(i)}$-s and $F_{\alpha,\delta}$ remains non-ambiguous. The benefit of such a choice of K_C is that the order of the modification around the points of $\mathscr{P}_i$ in the ith step can be arbitrary.

The first restriction states that the radius r_i of an interval centered at a point of $\mathscr{P}_i$ must be such that $r_i + \varepsilon_i < C^i$ for some $\varepsilon_i > 0$. Thus the choice $r_i = C^i \min\{C, 1 - 2C\}$ is appropriate.

The second restriction is that r_i must be smaller than the half of the minimal distance between two neighboring $\mathscr{P}_i$ points. More precisely, $r_i < \frac{C^{i-1} - 2C^i}{2}$, that is, $r_i = C^i \min\{C, 1 - 2C\}$ also works properly in this case.

6.1.3. *Hausdorff Spectrum of $F_{\alpha,\delta}$*

We defined f_G^F to be $-\infty$ (not 0) where f_G^F is not positive in Theorem 3.2.3.1 so that the scaling function $\tau(q)$ makes sense.

It is interesting to note, without proof, that the Hausdorff spectrum f_H^F of $F_{\alpha,\delta}$ consists of only two remarkable points $(1, 1)$ and (α, δ). Namely, $f_H^F(1) = 1$ and $f_H^F(\alpha) = \delta$. Furthermore, there may exist some pairs (β, γ) such that $1 \le \beta < \alpha$ if $\alpha > 1$ $(\delta \le \beta \le 1$ if $\alpha < 1)$ and $0 \le \gamma < (1 - \varepsilon)\delta$ such that for these pairs we have $f_H^F(\beta) = \gamma$ and for all remaining Hölder continuity exponents β, $f_H^F(\beta) = -\infty$.

6.2. The Formal Proof of Theorem 3.2.3.1

The proof is shown for any fixed α and δ, thus we write F instead of $F_{\alpha,\delta}$ in this section for simplicity.

The core of the proof is that we estimate the value of $N_n(\beta, \varepsilon)$ for all possible n, β, ε. (The value $N_n(\beta, \varepsilon)$ appears in the definition of the grained spectrum in Eq. (7).)

The exact proof consists of two similar parts. The first part deals with the case $\alpha > 1$ and the second with $1 > \alpha > \delta$. Since the function $\phi_t(x) = \mathrm{sign}(x - t)|x - t|^\alpha$ is different concerning its slope in t only some elementary estimations differ in these two parts. The method is the same, so we omit the proof of the case $1 > \alpha > \delta$.

For simplicity, we introduce some notations:

- Let $I_k^n := [k2^{-n}, (k+1)2^{-n})$ and for some interval $I = [a, b]$ define the difference $\Delta_I F := F(b) - F(a)$
- $\phi_t^*(x) = \mathrm{sign}(x - t)|x - t|^\alpha + F(t)$ is called the basis function centered at t.
- An interval is called i-level interval if it is one of the remaining intervals after the ith removal step. We remark that its length is C^i.
- Let $a(n)$ and $b(n)$ be two sequences in the form of $a(n) = e^{\alpha(n)}A(n)$ and $b(n) = e^{\beta(n)}B(n)$ where α, β, A, B are generalized polynomials in n with negative powers allowed. So $a(n)$ and $b(n)$ tend to either to 0 or ∞ as n tends to infinity. The exponential parts of $a(n)$ and $b(n)$ are equal, i.e., $a(n) =^{\log} b(n)$, if $\alpha = \beta$. $a(n) \le^{\log} b(n)$ if there exists a sequence $c(n)$ such that $a(n) =^{\log} c(n)$ and $c(n) \le b(n)$. This definition implies that if $a(n) \le b(n)$ then $b(n) + a(n) =^{\log} b(n)$.

Practically, these relations ignore the polynomial multipliers.

6.2.1. *Proof of the Case $\alpha > 1$*

Step 0: The following facts are simple consequences of the definition of F.

For arbitrary $i \in \mathbb{N}$ if $t \in \mathscr{R}_i$ then the increments of $f^{(i)}$s do not change on $[t, t + r_i]$ and decrease on $[t - r_i, t]$, i.e.,

$$F \restriction_{[t,t+r_i]} = \phi_t^* \restriction_{[t,t+r_i]}; \quad F \restriction_{[t-r_i,t]} \geq f_t^{(j+1)} \restriction_{[t-r_i,t]} \geq f_t^{(j)} \restriction_{[t-r_i,t]} \geq \phi_t^* \restriction_{[t-r_i,t]} \quad (23)$$

if $x \in [t, t + r_i]$ and $j \geq i$. Moreover,

$$\Delta_{[t,x]}F = \Delta_{[t,x]}\phi_t^*; \quad \Delta_{[x,t]}F \leq \Delta_{[x,t]}\phi_t^{(j+1)} \leq \Delta_{[x,t]}f_t^{(j)} \leq \Delta_{[x,t]}\phi_t^* \quad (24)$$

if $x \in [t - r_i, t]$.

Similar statement is true for $t \in \mathscr{L}$:

$$F \restriction_{[t,t+r_i]} \leq f_t^{(j+1)} \restriction_{[t,t+r_i]} \leq f_t^{(j)} \restriction_{[t,t+r_i]} \leq \phi_t^* \restriction_{[t,t+r_i]}; \quad \phi_t^* \restriction_{[t-r_i,t]} = F \restriction_{[t-r_i,t]} \quad (25)$$

if $t \in \mathscr{L}_i$, $j \geq i$, and

$$\Delta_{[x,t]}F = \Delta_{[x,t]}\phi_t^*; \quad \Delta_{[t,x]}F \leq \Delta_{[t,x]}f_t^{(j+1)} \leq \Delta_{[t,x]}f_t^{(j)} \leq \Delta_{[t,x]}\phi_t^* \quad (26)$$

if $x \in [t, t + r_i]$.

Observe that if $t \in \mathscr{R}_i$

$$\Delta_{I_{k_1}^n}F < \Delta_{I_{k_2}^n}F \quad (27)$$

for some $k_1 < k_2$ such that $I_{k_1}^n, I_{k_2}^n \subset [t, t + r_i]$. Similar result is also given for $t \in \mathscr{L}_i$.

In the sequel, the proof is organized as follows:

- Step 1 estimates $N_n^{\emptyset}(\beta, \varepsilon) := \#\{I_k^n : \beta - \varepsilon \leq \frac{\log_2 \Delta_{I_k^n}F}{-n} \leq \beta + \varepsilon, I_k^n \cap \mathscr{E}^* = \emptyset\}$ for all $1 \leq \beta \leq \alpha$.
- Step 2 estimates $N_n^{\cap}(\beta, \varepsilon) := \#\{I_k^n : \beta - \varepsilon \leq \frac{\log_2 \Delta_{I_k^n}F}{-n} \leq \beta + \varepsilon, I_k^n \cap \mathscr{E}^* \neq \emptyset\}$ for all $1 \leq \beta \leq \alpha$. In this part only the key steps of the proof are presented, some long, technical calculations are omitted.
- In Step 3 $N_n(\beta, \varepsilon)$ is estimated for any $\beta < 1$ and $\alpha < \beta$ $(1 < \alpha)$. Finally, we summarize the results of the previous steps and conclude the proof.

Step 1. First of all, remark that $N_n(\beta, \varepsilon) = \#\{I_k^n : (\beta - \varepsilon) \vee 1 \leq \frac{\log_2 \Delta_{I_k^n}F}{-n} \leq (\beta + \varepsilon) \wedge \alpha, I_k^n \cap \mathscr{E}^* = \emptyset\}$ because for all I_k^n and arbitrarily small positive η we have $1 - \eta \leq \frac{\log_2 \Delta_{I_k^n}F}{-n} \leq \alpha + \eta$ for n above a large enough threshold. The proof is omitted.

Next, three subcases are separated: (a) $1 < \beta < \alpha$; (b) $\beta = \alpha$; and (c) $\beta = 1$.

Case (a): $1 < \beta < \alpha$. For any fixed β there exists a real number ε such that $1 < \beta - \varepsilon < \beta + \varepsilon < \alpha$. We use two important estimations, which are straightforward consequences of Eq. (27).

1. If $t \in \mathscr{R}_i$ and $I_k^n \subset [t, t + r_i]$ then $\frac{\log_2 \Delta_{I_k^n} F}{-n}$ decreases as k increases. Moreover, if d is the distance between t and I_k^n then thanks to the Lagrange's theorem one has

$$\alpha d^{\alpha-1} 2^{-n} \le \Delta_{I_k^n} F \le \alpha (d + 2^{-n})^{\alpha-1} 2^{-n}. \tag{28}$$

This elementary result means that if f is a real function which is differentiable on the interval $[a, b]$ then for every subinterval $[c, d] \in [a, b]$ the following inequality holds

$$|f(c) - f(d)| \le \sup_{x \in [c,d]} |f'(x)|\ |c - d|.$$

In addition, if $t \in \mathscr{L}_i$ and $I_k^n \subset [t, t - r_i]$ then $\frac{\log_2 \Delta_{I_k^n} F}{-n}$ decreases as k decreases and by symmetry we have the same inequality as in Eq. (28).

2. If t is a right-hand side endpoint, (the estimations are similar in the case of left-hand side endpoints because of the symmetry of the construction), there exist a minimal distance $d_{\beta,n}$ and a maximal distance $D_{\beta,n}$ such that

$$\beta - \varepsilon \le \frac{\log_2 \Delta_{I_k^n} F}{-n} \le \beta + \varepsilon \quad \text{ or equivalently } 2^{-n(\beta-\varepsilon)} \ge \Delta_{I_k^n} F \ge 2^{-n(\beta+\varepsilon)}. \tag{29}$$

for all k: $I_k^n \subset (t + d_{\beta,n}, t + D_{\beta,n})$.

Using Eqs. (28) and (29) one gets the lower estimates $\left(d_{\beta,n}^l, D_{\beta,n}^l\right)$ and the upper estimates $\left(d_{\beta,n}^u, D_{\beta,n}^u\right)$ of $\left(d_{\beta,n}, D_{\beta,n}\right)$:

$$d_{\beta,n}^u = \frac{1}{\alpha} 2^{-n\frac{\beta-1+\varepsilon}{\alpha-1}}, \quad D_{\beta,n}^u = \frac{1}{\alpha} 2^{-n\frac{\beta-1-\varepsilon}{\alpha-1}}, \quad d_{\beta,n}^l = d_{\beta,n}^u - 2^{-n}, \quad D_{\beta,n}^l = D_{\beta,n}^u - 2^{-n} \tag{30}$$

with $\varepsilon > 0$, $\beta - 1 - \varepsilon > 0$ ($d_{\beta,n}$ might be 0).

The proof continues with the estimation of $N_n^{\emptyset}(\beta, \varepsilon)$.

The lower estimate of $N_n^{\emptyset}(\beta, \varepsilon)$. For fixed n let $i_l(n)$ be the largest i for which the following condition is satisfied

$$t \in \mathscr{R}_i : \left(t + d_{\beta,n}, t + D_{\beta,n}^u\right) \subset (t, t + r_i).$$

Because of symmetry we get the same number if we consider the sets $\mathcal{L}_i$ instead of $\mathcal{R}_i$ in this condition. Let $\mathcal{E}_{i_l(n)}$ be the set of endpoints such that this property holds independently whether they are in $\mathcal{R}$ or $\mathcal{L}$.

Define the following set of intervals

$$\begin{aligned}&\left\{I_k^n : I_k^n \subset \left(t + d^u_{\beta,n}, t + D^l_{\beta,n}\right) \text{ if } t \in \mathcal{E}_{i_l(n)} \cap \mathcal{R}\right\}\\ &\quad \cup \left\{I_k^n : I_k^n \subset \left(t - D^l_{\beta,n}, t - d^u_{\beta,n}\right), \text{ if } t \in \mathcal{E}_{i_l(n)} \cap \mathcal{L}\right\}.\end{aligned}$$

By the definitions of $d_{\beta,n}, d^u_{\beta,n}, D^l_{\beta,n}, D^u_{\beta,n}, \mathcal{E}_{i_l(n)}$, any I_k^n interval of this set fulfills the property

$$\beta - \varepsilon \le \frac{\log_2 \Delta_{I_k^n} F}{-n} \le \beta + \varepsilon, \tag{31}$$

which results that the number of these intervals estimates $N_n^{\emptyset}(\beta, \varepsilon)$ from below. The more detailed calculation is shown below.

First, we have $i_l(n) = \max\{i : C^i K_C > D^u_{\beta,n}\}$. By Eq. (30), $i_l(n) = \left\lfloor n \frac{-1}{\log_2 C} \frac{\beta-1-\varepsilon}{\alpha-1} \right\rfloor + c_l = \left\lfloor n\delta \frac{\beta-1-\varepsilon}{\alpha-1} \right\rfloor + c_l$ for an appropriate constant c_l. Hence $\#\mathcal{E}_{i_l(n)} = 2^1 + 2^2 + \cdots + 2^{i_l(n)} \stackrel{\log}{=} 2^{n\delta \frac{\beta-1-\varepsilon}{\alpha-1}}$.

Besides, if $t \in \mathcal{E}_{i_l(n)} \cap \mathcal{R}$ on the interval $\left[t + d^u_{\beta,n}, t + D^l_{\beta,n}\right]$ or $t \in \mathcal{E}_{i_l(n)} \cap \mathcal{L}$ on $\left[t - D^l_{\beta,n}, t - d^u_{\beta,n}\right]$ we count $N_n(t) = N_n = \left(D^l_{\beta,n} - d^u_{\beta,n}\right) : 2^{-n} = \frac{1}{\alpha} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}} \left(1 - 2^{-n\frac{2\varepsilon}{\alpha-1}}\right) - 1 \stackrel{\log}{=} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}}$ intervals with length 2^{-n}.

Summarizing these two estimations we get

$$N_n^{\emptyset}(\beta, \varepsilon) \ge \sum_{t \in \mathcal{E}_{i_l(n)}} N_n(t) = \#\mathcal{E}_{i_l(n)} N_n \stackrel{\log}{=} 2^{n\delta \frac{\beta-1-\varepsilon}{\alpha-1}}\ 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}}. \tag{32}$$

The upper estimate of $N^n(\beta, \varepsilon)$. Basically, we follow the procedure described in the lower estimation above. Taking the points $t \in \mathcal{E}^*$ such that $|I(t)|/2 \ge d^l_{\beta,n}$ we have estimated from above the number of the possible points $t \in \mathcal{E}^*$ such that $\left[t + d^l_{\beta,n}, t + D^u_{\beta,n}\right] \subset I(t)$ or $\left[t - D^u_{\beta,n}, t - d^l_{\beta,n}\right] \subset I(t)$. Denote the set of these points by $\mathcal{E}_{i_u(n)}$ where $i_u(n) = \max\{i : C^i K_C \ge d^l_{\beta,n}\}$.

The idea of the upper estimation is that we consider the basis functions $\phi_t^*(x)$ centered at t, $t \in \mathcal{E}^*$, as if they do not change on both the left-hand and the right-hand side of the interval $I(t)$ during the later refinements. Virtually, it means that we also involve those intervals in the calculation that may be cut out in some later refinement (that is, the procedure overwrites them). Thus, we get upper estimation because each interval I_k^n which fulfills Eq. (31) may have been counted for more than one $t \in \mathcal{E}_{i_u(n)}$.

We obtain the upper estimation for the number of intervals in $\left[t+d^l_{\beta,n}, t+D^u_{\beta,n}\right]$ or $\left[t-D^u_{\beta,n}, t-d^l_{\beta,n}\right]$ if we take

$$N_n(t) = N_n = (D^u_{\beta,n} - d^l_{\beta,n}) : 2^{-n} = \frac{1}{\alpha} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}} \left(1 - 2^{-n\frac{2\varepsilon}{\alpha-1}}\right) + 1 =^{\log} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}}$$

intervals with length 2^{-n} in one branch of a basic function centered at some point of $\mathscr{E}_{i_u(n)}$.

Using Eq. (30), easy calculation shows that $i_u(n) = \left\lfloor n\delta\frac{\beta-1+\varepsilon}{\alpha-1}\right\rfloor + c_u$ for an appropriate constant c_u. Therefore $\#\mathscr{E}_{i_u(n)} = 2^1 + 2^2 + \cdots + 2^{i_u(n)} =^{\log} 2^{n\delta\frac{\beta-1+\varepsilon}{\alpha-1}}$.

It follows from the above discussion that

$$N_n^{\emptyset}(\beta,\varepsilon) \le \sum_{t\in\mathscr{E}_{i_u(n)}} 2N_n(t) = \#\mathscr{E}_{i_u(n)}\ 2N_n =^{\log} 2^{n\delta\frac{\beta-1+\varepsilon}{\alpha-1}}\ 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}} \tag{33}$$

From Eq. (32) and Eq. (33) one gets

$$2^{n\delta\frac{\beta-1-\varepsilon}{\alpha-1}} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}} \le^{\log} N_n^{\emptyset}(\beta,\varepsilon) \le^{\log} 2^{n\delta\frac{\beta-1+\varepsilon}{\alpha-1}} 2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}}. \tag{34}$$

Case (b). $\beta = \alpha$. Similar proof as the previous case results Eq. (34) with $\beta = \alpha$ and the change $d_{\beta,n} = 0$.

Case (c). $\beta = 1$. Define the following set of intervals $\mathscr{I}^n(1) = \{I^n_k \in \mathscr{I}^n : d(I^n_k, \mathscr{E}) > 2^{-n-1}\}$. Put $\{I^n_k \in \mathscr{I}^n(1) : I^n_k \subset \bigcup \mathscr{I}^m(1)\}$ for any n such that $m < n$. This is the set of intervals in $\mathscr{I}^n(1)$ which are also included in $\bigcup \mathscr{I}^m(1)$. For the elements of this set we have the estimation

$$(2^{-m-1})^{\alpha-1} 2^{-n} = \left(\min\left\{d(I^n_k, \mathscr{E}) \mid I^n_k \subset \bigcup \mathscr{I}^m(1)\right\}\right)^{\alpha-1} 2^{-n} \le \Delta_{I^n_k} F \le 2^{-n} \tag{35}$$

using the Lagrange inequality and the trivial estimation on the right. Thus for these intervals we have

$$1 \le \frac{\log_2 \Delta_{I^n_k} F}{-n} \le 1 + \frac{m+1}{n}(\alpha - 1).$$

Fixing $\varepsilon > 0$ and taking n such that $\frac{m+1}{n}(\alpha-1) \le \varepsilon$ we have $N_n^{\emptyset}(1,\varepsilon) \ge \#\mathscr{I}^m(1) 2^{n-m}$. Using this estimate and the trivial estimation $N_n^{\emptyset}(1,\varepsilon) \le 2^n$ one obtains

$$N_n^{\emptyset}(1,\varepsilon) =^{\log} 2^n. \tag{36}$$

Step 2. The following will be proved:

$$N_n^{\cap}(\alpha,\varepsilon) =^{\log} 2^{n\delta} \quad \text{and} \quad N_n^{\cap}(\beta,\varepsilon) \le^{\log} 2^{n\delta} \text{ for } 1 \le \beta < \alpha. \tag{37}$$

Let $\mathcal{I}^n(\alpha) = \{I_k^n \in \mathcal{I}^n : I_k^n \cap \mathcal{E} \neq \emptyset\}$. We provide a lower and some upper estimations of $\Delta_{I_k^n} F$ if $I_k^n \in \mathcal{I}^n(\alpha)$.

The lower estimate of $\Delta_{I_k^n} F$. Define the *level of minimal cover* for arbitrary integer n: $i(n) = \min\{i : 2^{-n} K_C > C^i\}$. Hence, $i(n) = \lfloor n\delta \rfloor + \kappa_C$ where κ_C is a suitable constant.

If $I_k^n \cap \mathcal{E}^* \neq \emptyset$ then it is easy to show that I_k^n intersects at least one $i(n)$-level interval. Hence, $I_k^n \cap \mathcal{E}_{i(n)} \neq \emptyset$. Let $t_k^n \in \mathcal{E}_{i(n)}$ be the endpoint for which $t_k^n \in I_k^n$. If there is more than one endpoint with this property one of them is chosen arbitrarily.

Since $r_i \leq r_j$ if $i \geq j$, the choice of t_k^n and the definition of $i(n)$ implies that $\left[t_k^n - r_{i(n)}, t_k^n\right) \subset I_k^n$ or $\left[t_k^n, t_k^n + r_{i(n)}\right) \subset I_k^n$. Therefore, if $I_k^n \in \mathcal{I}^n(\alpha)$ we have

$$\min\left\{\Delta_{[t_k^n - r_i, t_k^n)} F, \Delta_{[t_k^n, t_k^n + r_i)} F\right\} \leq \Delta_{I_k^n} F.$$

We can estimate both terms on the left from below, the lower estimation of $\Delta_{I_k^n} F$ is of the form:

$$\Delta_{I_k^n} F \geq^{\log} C^{i(n)\alpha} =^{\log} 2^{-n\alpha}. \tag{38}$$

The upper estimate of $\Delta_{I_k^n} F$. Let $i^*(n) = \max\left\{i : 2K_C 2^{-n} < C^i\right\}$ and define the set $\mathcal{I}^n(\alpha)^* = \left\{I_k^n : \text{ there exists a point } t_{I_k^n} \in \mathcal{P}_{i^*(n)} \text{ such that } I_k^n \subset I(t_{I_k^n})\right\}$. If $I_k^n \in \mathcal{I}^n(\alpha)^*$, we have a trivial upper estimation using Eqs. (23) and (24) and Eqs. (25) and (26):

$$\Delta_{I_k^n} F \leq \sup_{x \in I_k^n} \left|x - t_{I_k^n}\right|^\alpha \leq \sup_{x \in I(t_{I_k^n})} \left|x - t_{I_k^n}\right|^\alpha = (C^{i^*(n)})^\alpha \leq c 2^{-n\alpha} \tag{39}$$

for some constant c. On $\mathcal{I}^n(\alpha) \backslash \mathcal{I}^n(\alpha)^*$ we also have the trivial estimate $\Delta_{I_k^n} F \leq c 2^{-n\alpha}$.

Summarizing the results on the lower and the upper estimates we get the following: if $I_k^n \in \mathcal{I}^n(\alpha)^*$ then

$$\frac{\log_2 \Delta_{I_k^n} F}{-n} =^{\log} 2^{-n\alpha},$$

and if $I_k^n \in \mathcal{I}^n(\alpha) \backslash \mathcal{I}^n(\alpha)^*$ then

$$\frac{\log_2 \Delta_{I_k^n} F}{-n} \leq^{\log} 2^{-n}.$$

Therefore, the estimations of $N_n^{\cap}(\alpha,\varepsilon)$ and $N_n^{\cap}(\beta,\varepsilon)$ for $\beta<\alpha$ are the following

$$\begin{aligned} 2^{n\delta} &=^{\log} \#\mathcal{I}^n(\alpha)^* \le N_n^{\cap}(\alpha,\varepsilon) \le \#\mathcal{I}^n(\alpha) =^{\log} 2^{n\delta} \\ N_n^{\cap}(\beta,\varepsilon) &\le \#\mathcal{I}^n(\alpha)\backslash\mathcal{I}^n(\alpha)^* \le \#\mathcal{I}^n(\alpha) =^{\log} 2^{n\delta} \quad \text{for } \beta<\alpha. \end{aligned} \tag{40}$$

Step 3. Recall that $N_n(\beta,\varepsilon)=N_n^{\emptyset}(\beta,\varepsilon)+N_n^{\cap}(\beta,\varepsilon)$ for all possible β,ε,n. Using Eqs. (34), (36), (37), and (40) one gets

$$N_n(\beta,\varepsilon) =^{\log} \begin{cases} 2^n & \text{if } \beta=1, \\ 2^{n\delta\frac{\beta-1+\varepsilon}{\alpha-1}}2^{n\frac{\alpha-\beta+\varepsilon}{\alpha-1}} & \text{if } 1<\beta<\alpha, \\ 2^{\delta n} & \text{if } \beta=\alpha. \end{cases} \tag{41}$$

Taking $f_G^F(\beta)=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{\log_2(N_n(\beta,\varepsilon))}{n}$ we get $f_G^F(\beta)=1-(1-\delta)\frac{\beta-1}{\alpha-1}$ for $1\le\beta\le\alpha$.

In the previous steps we have proved that $2^{-n\alpha}\le^{\log}\Delta_{I_k^n}F\le^{\log}2^{-n}$ for all I_k^n. Therefore, if $\beta\notin[1,\alpha]$

$$f_G^F(\beta)=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{\log_2(N_n(\beta,\varepsilon))}{n}=-\infty$$

as it was required.

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