Performance Analysis of Multifractal Network Traffic

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Abstract. In this paper we present some new results in the framework of multifractal performance analysis including both the characterization and modeling of multifractal network traffic, and also the multifractal queueing performance analysis. We first propose a new multifractal traffic model for network traffic based on the combination of a multiplicative cascade with an independent lognormal process. This traffic model is able to provide a very accurate fit to the multifractal characteristics of data traffic including both the scaling function and the moment factor, but it is also simple enough from a practical point of view having only three parameters. In addition, the model features many important properties observed in data traffic including LRD, multifractality and lognormality.

We also present an approximation for the queue tail asymptotics in an infinite capacity single server queue serviced at a constant rate driven by a general multifractal input process. We show that in the special and important case of the monofractal fractional Brownian motion (fBm) input traffic our result gives the well-known Weibullian tail. We prove that the class of Gaussian processes with scaling properties is in the class of monofractal processes and we derive the related characterization functions. Applying the approximation we provide a new practical method for queueing performance estimation of general multifractal traffic.

Finally, we present a practical case study to show the practical application of our framework for measured data traffic and also to validate both our multifractal model and our queueing results.

1 Introduction

It has widely been recognized that network traffic is extremely bursty. Particularly, the *fractal* nature of the investigated LAN/WAN traffic is of special interest [1, 2, 3, 4, 5, 6, 7, 8, 9]. This surprising *scaling phenomenon* observed in data traffic is that the burstiness of the traffic are presented over several time scales [1].

In the research of this phenomenon first, *long-range dependence* (LRD) and *self-similarity* have been identified and a number of studies is concentrated on how to detect accurately the LRD property and how to estimate the Hurst parameter [10, 11]. The LRD property of a traffic flow is revealed in the power law decay of the autocorrelation function at large lags, i.e.,

$$r(k) \sim c|k|^{2H-2}, \qquad k \to \infty,\ H \in (0.5, 1),$$

where c is a constant [10]. The degree of this slow decay is determined by the Hurst parameter (H).

Fractal traffic models (fractional Brownian motion (fBm) models, FARIMA models, Cox's M/G/∞ models, on/off models, etc.) to capture LRD and self-similar properties have also been developed [1, 12, 13, 9]. Among these models the fBm [14] was found to be a popular parsimonious and tractable model of traffic aggregation. It was shown that the fBm is an accurate model if the traffic is aggregated from a large number of independent users whose peak rates are small relative to link capacity and the flow control has no significant impact, see [15, 16, 17, 18, 19] for more details. The performance implications of the fractal property are also addressed in a series of studies [20, 21].

However, in the fractal research of network traffic it was also discovered that the LAN/WAN traffic has a more complex scaling behavior which cannot be described by LRD and self-similarity [2, 4, 5]. More precisely, it has been found that the aggregate network traffic is asymptotically self-similar over time scales on the order of few hundreds of milliseconds and above but it exhibits *multifractal* scaling below this time scale. It has been also pointed out that the transition from the multifractal to self-similar scaling occurs around time scales of a typical packet round-trip time in the network [4, 5]. Moreover, some studies showed that multifractal scaling can also be present even at large time scales [22].

A stochastic process $X(t)$ is called *multifractal* [23, 24] if it has stationary increments and satisfies $\mathbb{E}[|X(t)|^q] = c(q)t^{\tau(q)+1}$ for some positive q, where $\tau(q)$ is called the *scaling function* of multifractality and $c(q)$ is independent of t. If the scaling function $\tau(q)$ is a linear function of q the process is called *monofractal*. Multifractal processes are also called processes with *scaling property*. Multifractality is thus defined as a *global* property of the process moments. Multifractality can capture more "irregularities" in the scaling behavior as we consider higher-order characteristics of the traffic. As an example, if we characterize a self-similar process (e.g. fractional Brownian motion) we can only consider the second-order characteristics, i.e., the autocorrelation function. The characterization of a general multifractal process by the autocorrelation function is not adequate and we need to investigate higher-order statistics too. For this reason the monofractal traffic models (e.g. fBm) are inadequate to characterize the network traffic with multifractal properties. An interesting exception is the model presented in [25], which is still a monofractal model but it is able to capture high-order statistics, so it has a more flexible modeling ability compared to self-similar traffic models. The physical explanations and engineering implications are also addressed in several papers [5, 8].

The use of multifractals in network traffic modeling is a new topic and in spite of the fact that some models are already published, the complete understanding of this phenomenon and the application of the models in practice is far from being complete.

There are different processes which are candidates for multifractal modeling. *Multiplicative cascades* were first used as a multifractal model for data traffic [26, 6]. This class is the most well-known member of the class of multifractal processes. The simplest case of this process is the *binomial cascade* which can be defined by a binary tree structure [24, 26]. Combining this process with the aforementioned fBm we can define a new class called the *fractional Brownian motions in multifractal time* [27]. This process has several nice properties, e.g. it is able to capture LRD and multifractal scaling independently. The *self-similar α-stable process* [28] is a different multifractal process. Its statistics of order $q \geq \alpha$ are not finite resulting in an irregular multifractal structure. One of the simplest process from this class is the *linear fractional stable motion*.

In this paper we propose a new multifractal model which is based on the pairwise product of a multiplicative cascade and an independent lognormal process. Earlier published multifractal models based on multiplicative cascades do not aim to capture the complete multifractal characteristics (including both the scaling function and the moment factor) and/or using many parameters for the model [7, 6, 29, 30, 31]. Comparing these models to our new model we argue that our model is simple enough for practical purposes having only three parameters but, on the other hand, flexible enough to capture accurately the whole multifractality of the traffic. We show the statistical properties of the model and also the applications for actual measured network traffic. The model is validated by comparing the queueing performance produced by the original measured traffic to the queueing results obtained by the traffic generated by our model in the presented case study.

From a practical point of view queueing analysis of fractal traffic is a very important issue for network dimensioning and management. Therefore the study of queueing systems with fractal traffic input is a challenge in queueing theory. In the recent years the performance of queues with LRD or self-similar input has been deeply analyzed [32, 33, 34, 1]. A collection of studies has proven that the fBm based models have a tail queue distribution that decays asymptotically like a Weibullian law [1]

$$\mathbf{P}[Q > b] \simeq \exp(-\delta b^{2-2H}) \qquad (1)$$

where δ is a positive constant that depends on the service rate of the queue [14, 35]. This important result shows that queues with fBm input ($H > 1/2$) have a much slower decay than that of the exponential.

However, there is a lack of queueing results available in the cases when the input traffic has a more complex scaling behavior. Especially, queueing systems with multifractal input are an undiscovered field and only a few results published in the literature. Véhel *et al.* [36] suggested a cascade model for TCP traffic based on the retransmission and congestion avoidance mechanisms with no performance analysis. Riedi *et al.* [7] developed a multiscale queueing analysis in the case of tree-based multiscale input models. Gao *et al.* simulated queues fed by multiplicative multifractal processes in [37] but provided no analytical results. In contrast to these results we consider *general multifractal process* without any restrictions and derive an *analytical approximation for the queue tail asymptotics*.

In this paper, besides the mentioned new model, we also present a performance analysis of multifractal queues. Our aim is to contribute to the queueing theory of multifractal queues and also to the traffic engineering implications. We give an approximation formula for the tail of the queue with multifractal traffic input. The formula results in the asymptotically Weibullian decay for the special monofractal case of fractional Brownian motion input process, which is consistent with the previously mentioned results [14, 35]. In this framework we show that any Gaussian process with scaling properties is in the class of monofractal processes and derive the related characterization functions. We also consider the special case when the queue is driven by a Gaussian process with independent, stationary increments (Brownian motion) and show that our result is in a good agreement with results known from the theory of Gaussian processes.

The paper demonstrates the application of our results (both the new model and our queueing approximation) in a case study for some measured WAN traffic.

The rest of the paper is organized as follows. Section II overviews the foundations of the multifractal concept and a simple method for multiscaling functions estimation. Section III presents our new multifractal traffic model with its statistical properties. Our approximation for queue tails with applications is described in Section IV. Section V demonstrates the practical applicability of the whole multifractal framework in a case study from modeling of measured network traffic to its queueing analysis. Finally, Section VI concludes the paper and presents our targets for future research directions.

2 MULTIFRACTAL TRAFFIC ANALYSIS

In this section the basic concepts of multifractals are briefly discussed. A simple method based on the absolute moments for testing of the scaling properties and for estimation of the multifractal characteristics is also presented.

2.1 MULTIFRACTAL CONCEPTS

The multifractal concept was first introduced by Mandelbrot in the context of turbulence in the early 70's. Since then multifractal processes have been widely used in a variety of research fields like geophysics, image processing, stock market modeling, and recently network traffic characterization.

A stochastic process $X(t)$ is called multifractal if it has stationary increments and satisfies [24]

$$\mathbb{E}(|X(t)|^q) = c(q)t^{\tau(q)+1} = c(q)t^{\tau_0(q)}, \qquad (2)$$

for some positive values $q \in Q$, $[0,1] \subset Q$, where $\tau(q)$ is called the *scaling function* and the *moment factor* $c(q)$ is independent of t. In this paper we refer to $\tau_0(q)$ as the scaling function instead of $\tau(q)$. An easy consequence of the definition is that $\tau(q)$ is a concave function [24]. If $\tau(q)$ is linear in q the process is called monofractal, otherwise it is multifractal. It can be shown that in the special case of self-similar process with index H we get $\tau(q) = qH - 1$ and $c(q) = \mathbb{E}(|X(1)|^q)$. The most obvious examples of multifractals are self-similar and multiplicative processes.

The definition above describes multifractality in terms of process moments and it may lead to a more intuitive understanding of multifractality. However, there is an alternative approach to multifractals, also found in the literature, which is based on the study of the local erratic behavior of the process by means of its local Hölder exponents. For the details of this approach, see [26] and references therein.

Without loss of generality consider a process $X(t)$ defined on $I = [0,1]$. For any n this interval can be decomposed as the disjoint union of 2^n dyadic subintervals $I_n^k = [k2^{-n}, (k+1)2^{-n})$, where $k = 0, 1, \ldots, 2^n - 1$. The *coarse Hölder exponents* at resolution n are defined as the rescaled logarithmic increments of the process on I_n^k, i.e., $\alpha_n^k = -\frac{1}{n}\log_2 |X((k+1)2^{-n}) - X(k2^{-n})|$. Define $k_n(t)$ such that $I_n^{k_n(t)}$ is the unique interval which contains t. The *local Hölder exponent* of X at t is defined as

$$\alpha(t) = \liminf_{n\to\infty} \alpha_n^{k_n(t)} \qquad (3)$$

Multifractal analysis consists in classifying the local irregularities of the process and "measuring" the set of points with the same degree of irregularity. This measurement may be done in several different ways. One approach relies on the fact that points with equal local Hölder exponents lie on certain fractal sets whose Hausdorff dimension can be, at least theoretically, computed (*Hausdorff spectrum*). A second approach proceeds by simply counting the coarse exponents at resolution n sufficiently close to the given Hölder exponent and taking appropriate limits of the rescaled logarithmic number thus obtained. This method is closely related to the probabilistic law of large numbers and large deviation theory, yielding the *grained* or *large deviation spectrum*. A third approach, best suited for applications, uses the so-called *partition functions*, which can be derived from the moments of the process. The spectral function such defined is termed the *Legendre spectrum*, since under suitable conditions it is the familiar Legendre transform of the grained spectrum. These definitions are interrelated by properties referred to as different forms of the so-called *multifractal formalism*, i.e.,

$$f(\alpha) = \inf_q(\alpha q - \tau(q)), \qquad (4)$$

where $f(\alpha)$ denotes the multifractal spectrum. Our definition of multifractality gives direct access to this latter definition of the spectrum.

2.2 ESTIMATION OF MULTISCALING FUNCTIONS

The full description of a multifractal model involves both the moment factor $c(q)$ and the scaling function $\tau(q)$. We present here a simple method for testing of scaling properties and also for the estimation of these functions.

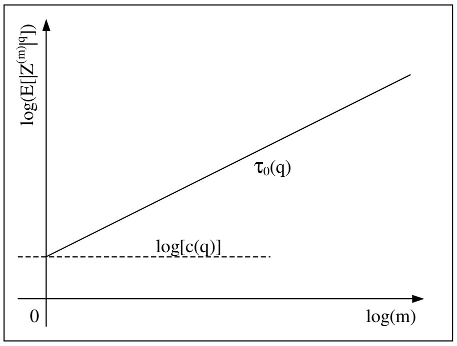

Figure 1: A simple method for scaling test and the estimation of $c(q)$ and the scaling function $\tau(q)$.

The definition of multifractal processes (Eq. 2) claims the stationarity condition for the increments. Therefore it is easy to verify the following relation for the moments of the increments:

$$\mathbb{E}[|Z^{(\triangle t)}|^q] = c(q)(\triangle t)^{\tau(q)+1} = c(q)(\triangle t)^{\tau_0(q)}, \quad q > 0, \qquad (5)$$

where $Z^{(\triangle t)}$ denotes the increment process of time sample $\triangle t$. Thus this equality also holds for $m = 1, 2, \ldots$

$$\mathbb{E}[|Z^{(m\triangle t)}|^q] = c(q)(m\triangle t)^{\tau_0(q)}, \qquad q > 0. \quad (6)$$

Choose $\triangle t$ as the time unit, then

$$\log \mathbb{E}[|Z^{(m)}|^q] = \tau_0(q)\log m + \log c(q), \qquad q > 0. \quad (7)$$

Based on this property, the method is the following: Given a data series of a process increments $Z_1, Z_2, \ldots, Z_n$ and define its corresponding *real* aggregated sequence $\{Z^{(m)}\}$ of the aggregation level m by

$$Z_k^{(m)} = \sum_{i=1}^{m} Z_{(k-1)m+i}, \qquad k, m = 1, 2, \ldots \quad (8)$$

If the sequence $\{Z_k\}$ has scaling property then the plot of absolute moments $\mathbb{E}[|Z^{(m)}|^q]$ versus m on a log-log plot should be a straight line due to Eq. 7. The slope of the straight line provides the estimate of $\tau_0(q)$ and the intercept is the value for $\log c(q)$. The illustration of the method can be seen in Fig. 1.

3 A Multifractal Cascade Model

This section presents our multifractal model based on the multiplicative cascade. We show the detailed construction of the model and the procedure to estimate model parameters from a real data trace. Comparison to other multifractal models is also discussed.

3.1 Multiplicative cascade

The simplest multifractals are typically constructed by an iterative procedure called multiplicative cascade. Consider a unit interval associated with a unit mass. At state $k = 1$ divide the unit interval into two equal subintervals and associate with them the mass r and $1 - r$, respectively. The fraction r is called the *multiplier*. The same rule is applied to each subinterval and its associated mass. The iterative procedure of cascade construction is shown in Fig. 2. The multipliers r are chosen to be independent random variables R concentrated on $[0, 1]$ with the probability distribution function $F_R(x)$, $\mathbb{E}(R) = 1/2$. We also choose the multiplier r to have a symmetric density function so that r and $1 - r$ have the same marginal distribution. Thus at the state k a dyadic interval of length $\triangle t_k = 2^{-k}$ starting at $t = 0.\eta_1 \ldots \eta_k = \sum \eta_i 2^{-i}$ has the mass (measure)

$$\mu(\triangle t_k) = R(\eta_1)R(\eta_1, \eta_2)\ldots R(\eta_1, \ldots, \eta_k),$$

where $R(\eta_1, \ldots, \eta_i)$ indicates the multiplier at state number i. Since multipliers are i.i.d. it is easy to show that the measure μ satisfies the scaling relationship:

$$\mathbb{E}(\mu(\triangle t_k)^q) = (\mathbb{E}(R^q))^k = \triangle t_k^{-\log_2 \mathbb{E}(R^q)},$$

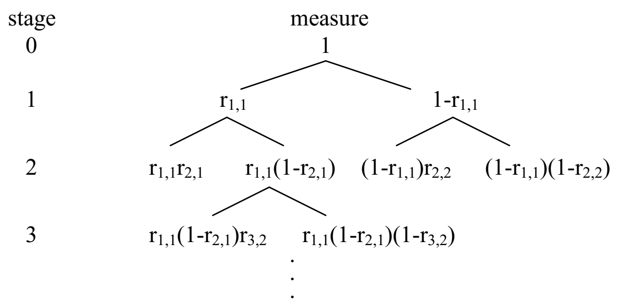

Figure 2: The iterative procedure of cascade construction.

which defines a multifractal process with scaling function $\tau_0(q) = -\log_2 \mathbb{E}(R^q)$.

Note that the multifractal process constructed above is also referred to as *conservative cascade*. An important property of this random cascade is its dependence structure due to the construction. If the multipliers used in the construction have the same fixed value r_0 $(0 < r_0 < 1)$ then the obtained multiplicative measure is called *binomial*. The binomial measure is a deterministic cascade, its scaling function being $\tau_0(q) = -\log_2(r_0^q + (1 - r_0)^q) + 1$. In addition, if the iteration only conserves mass on the average,i.e., multipliers at each mass division are also i.i.d. but have mean of 1/2, the corresponding measure is called *canonical* [24]. An example of the multiplicative cascades can be seen in Fig. 3.

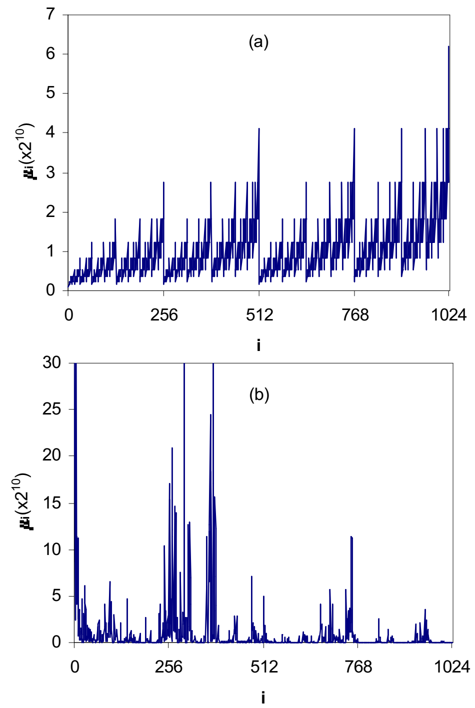

Figure 3: Multiplicative measures at stage $k = 10$: (a) binomial cascade ($r_0 = 0.4$), (b) conservative cascade ($R \sim Uniform(0, 1)$).

From the network modeling point of view only conservative cascades are of interest. The binomial cascade is naturally excluded because it is a deterministic process. The canonical cascade cannot be used since it is an independent random process, while network traffic flows are long-range dependent. In this study we use the conservative cascade as a building block of our traffic model.

3.2 A MULTIFRACTAL TRAFFIC MODEL

Our new traffic model is built on a combination of a multiplicative cascade and a lognormal process. The motivation of this construction is to get a flexible traffic model which is able to capture the multifractal characteristics of network traffic characterized by not only the scaling function but also by the moment factor.

Suppose that the multifractal analysis of a real data series obtained form measured network traffic shows its multifractal properties characterized by the scaling function $\tau_0(q)$ and the moment factor $c(q)$. The obvious task of cascade modeling is to find a convenient probability distribution for multipliers R such that $-\log_2(\mathbb{E}(R^q)) = \tau_0(q)$. However, this cascade model captures only the multifractal properties given by the scaling function and fails to furnish any information on the moment factor $c(q)$. The idea for a more comprehensive traffic model is the following: 2^N synthetic data $\mu(\triangle t_N)$ are first generated by a multiplicative cascade with multipliers governed by the distribution of R. Then we generate a data series of length 2^N by forming the pairwise product of the cascade data and an i.i.d. sample of a positive random variable Y with the same length. The variable Y is chosen to be independent of the cascade measure $\mu(\triangle t_N)$, thus the obtained series, denoted by $X(\triangle t_N)$, satisfies

$$\mathbb{E}(X(\triangle t_N)^q) = \mathbb{E}(Y^q)\mathbb{E}(\mu(\triangle t_N)^q) = \mathbb{E}(Y^q)\triangle t_N^{\tau_0(q)}. \tag{9}$$

The model fitting task is to find the suitable random variable R and Y such that

$$\begin{cases} -\log_2(\mathbb{E}(R^q)) &= \tau_0(q) \\ \mathbb{E}(Y^q) &= c(q). \end{cases}$$

The presented model is relevant to multifractal network traffic due to, at least, the following reasons. First, it captures not only the multifractal properties of the traffic, but also fits some moments of the measured traffic data, which is the main motivation of our model. Second, the model, as the combination of multifractal measure $\mu(\triangle t_N)$ and a random variable Y, can fulfill the structural mechanics of the network traffic. The multifractal measure represents the operating mechanics of TCP protocols which leads to multifractal traffic data at packet level, e.g., a web session generate requests, each request gives rise to connections, each connection is made up of flows, flows consist of individual packets [5]. In addition, the variable Y gives the randomness to the data flows caused by, e.g., requested file sizes, actual network conditions, etc. Because of the multiplicative structure the model traffic has approximately lognormal distribution, see section 3.4 for details, which is in agreement with recent analysis study of wide-area TCP traffic [38].

For practical use, we introduce some modifications to the model. The measure $\mu(\triangle t_N)$ has a very small value since it is the product of N multipliers $0 < r < 1$, so to avoid loss of information we multiply the cascade measures by 2^N. Since $\mathbb{E}(\mu(\triangle t_N)) = 2^{-N}$, this normalizes the cascade increment so that it has unit mean. As another modification, we also rescale the cascade process to have unit time interval at stage N instead of $\triangle t_N = 2^{-N}$. For a multifractal increment $X_\triangle$

$$\begin{aligned} \mathbb{E}(X_\triangle^q) &= c_1(q)\triangle t_1^{\tau_0(q)} \\ &= c_1(q)\left(\frac{\triangle t_2}{\triangle t_1}\right)^{-\tau_0(q)}\triangle t_2^{\tau_0(q)} \\ &= c_2(q)\triangle t_2^{\tau_0(q)}, \end{aligned} \tag{10}$$

which means that the choice of time unit influences the value of the moment factor $c(q)$. After applying these changes to the model we obtain

$$\mathbb{E}(X(\triangle t_0)^q) = \mathbb{E}(Y^q)2^{N[q+\log_2\mathbb{E}(R^q)]}\triangle t_0^{-\log_2\mathbb{E}(R^q)}, \tag{11}$$

where $\triangle t_0$ denotes the unit time interval of the data traffic to be modelled.

3.3 MODEL PARAMETERS

For multifractal traffic data, the scaling function $\tau_0(q)$ and the logarithm the moment factor $c(q)$ can be estimated by a simple absolute moment method, see [39] for details. Denote these estimated functions by $\tilde{\tau}_0(q)$ and $\log\tilde{c}(q)$, respectively. Owing the modifications mentioned above, the random variable R and Y should be chosen such that

$$-\log_2(\mathbb{E}(R^q)) = \tilde{\tau}_0(q) \tag{12}$$

$$\begin{aligned} \log\mathbb{E}(Y^q) &= \log\tilde{c}(q) - [q+\log_2\mathbb{E}(R^q)]N\log 2 \\ &= \log\tilde{c}(q) - [q-\tilde{\tau}_0(q)]N\log 2. \end{aligned} \tag{13}$$

Our analysis of various measured traffic with multifractal properties shows that the choice of R as a *symmetric beta random variable* on [0,1], Beta(α,α), with only one parameter $\alpha > 0$ is accurate to model the estimated scaling function. In this case

$$\tau_0(q) = \log_2\frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)}, \tag{14}$$

where $\Gamma(.)$ denotes the Gamma function.

Next, we choose the *lognormal distribution* for the random variable Y. It has two parameters m and σ and the moment is of the form $\mathbb{E}(Y^q) = e^{mq+\sigma^2q^2/2}$. Thus from Eq. 13 m and σ should be chosen such that

$$\begin{aligned} &mq + \frac{\sigma^2q^2}{2} = \\ &= \log\tilde{c}(q) - \left\{q - \log_2\frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)}\right\}N\log 2. \end{aligned} \tag{15}$$

In summary, our presented multifractal model has three parameters (α, m, σ) and the following characterization functions:

$$\begin{cases} \tau_0(q) &= \log_2 \frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)} \\ c(q) &= e^{mq+\sigma^2 q^2/2} 2^{N\left(q-\log_2 \frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)}\right)}. \end{cases}$$

Note that if the right-hand side of Eq. 13 is a concave function of q then the examined traffic data cannot be captured by our model since the absolute moment of any stochastic process is a *log-convex* function of the moment order q. (This property is easily derived from the Hölder inequality.) This kind of multifractal traffic cannot be fully characterized by a cascade based model.

It should be mentioned that the choice of the beta distribution for multiplier R and the lognormal distribution for random variable Y in this study is motivated by both their analysis simplicity and also the good fitting accuracy to the measured traffic data. However, for some different traffic data sets we may also find other choices which provide the accurate fit.

3.4 STATISTICAL PROPERTIES

Statistical properties of multiplicative cascades are studied in a number of papers, see [40, 30] for example. We apply and extend these properties for 2^N synthetic samples of our multifractal model.

(i) As an extension of the conservative cascade our traffic model is an exact *positive multifractal process*. In the model construction presented in the previous section, the multifractality is characterized by scaling function $\tau_0(q) = \log_2 \frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)}$ and the logarithm of the moment factor $\log c(q) = (mq + \sigma^2 q^2/2) + \left(q - \log_2 \frac{\Gamma(\alpha)\Gamma(2\alpha+q)}{\Gamma(\alpha+q)\Gamma(2\alpha)}\right) N \log 2$ where (α, m, σ) are model parameters.

(ii) The mean and the variance of the model process are: $\mathbb{E}(X(\triangle t_0)) = \mathbb{E}(Y) = e^{m+\sigma^2/2}$; $\mathrm{Var}(X(\triangle t_0)) = \mathbb{E}(Y^2)2^{2N}\mathbb{E}(R^2)^N - \mathbb{E}(X(\triangle t_0))^2 = e^{2m+2\sigma^2}(\frac{\alpha+1}{\alpha+1/2})^N - e^{2m+\sigma^2}$.

(iii) For $N \gg 1$, $X(\triangle t_0)$ has lognormal distribution. This property is deduced directly from the fact that $X(\triangle t_0) = 2^N.Y.R(\eta_1)\dots R(\eta_1,\dots,\eta_N)$ and the central limit theorem.

(iv) $X(\triangle t_0)$ has long-range dependent correlation structure.
Consider the covariance $\mathrm{cov}[X(\triangle t_0)_n, X(\triangle t_0)_{n+k}]$, where $k = 2^p, p = 1, 2, \dots$, which can be derived as follows:

$$\mathrm{cov}[X(\triangle t_0)_n, X(\triangle t_0)_{n+k}] = \mathbb{E}(Y)^2 \left\{2^{2N}\mathbb{E}(\mu(\triangle t_N)_n.\mu(\triangle t_N)_{n+k}) - 1\right\}. \quad (16)$$

The two measures $\mu(\triangle t_N)_n$ and $\mu(\triangle t_N)_{n+k}$ have the same ancestor at stage $N-p-1$, denoted by $\mu(\triangle t_{N-p-1})$, thus $\mu(\triangle t_N)_n = \mu(\triangle t_N)_{N-p-1}.r_{N-p}\prod_{i=N-p+1}^{N} r_{i,j_1}$ and $\mu(\triangle t_N)_{n+k} = \mu(\triangle t_N)_{N-p-1}(1-r_{N-p}).\cdot\prod_{i=N-p+1}^{N} r_{i,j_2}$ where $r_{i,j}$ denotes the actual multiplier values at stage i. Then $\mathbb{E}[\mu(\triangle t_N)_n.\mu(\triangle t_N)_{n+k}] = \mathbb{E}[\mu(\triangle t_{N-p-1})^2].\,.\mathbb{E}[r_{N-p}(1-r_{N-p})]\mathbb{E}[\prod_{i=N-p+1}^{N} r_{i,j_1}r_{i,j_2}] = \mathbb{E}(R^2)^{N-p-1}[\frac{1}{2} - \mathbb{E}(R^2)](1/2)^{2p}$. Insert this into Eq. 16 we get

$$\begin{aligned} &\mathrm{cov}[X(\triangle t_0)_n, X(\triangle t_0)_{n+k}] = \\ &= e^{2m+\sigma^2}\left\{\frac{\alpha(\alpha+1)^{N-1}}{(\alpha+1/2)^N}\left[\frac{\alpha+1}{\alpha+1/2}\right]^{-p} - 1\right\} \\ &= e^{2m+\sigma^2}\frac{\alpha(\alpha+1)^{N-1}}{(\alpha+1/2)^N}k^{-\log_2\left(\frac{\alpha+1}{\alpha+1/2}\right)} - e^{2m+\sigma^2}. \quad (17) \end{aligned}$$

Thus when N, k are large the covariance is ruled by $k^{-\log_2\left(\frac{\alpha+1}{\alpha+1/2}\right)}$, i.e., the model has LRD structure with Hurst parameter $H = 1 - \frac{\log_2 \frac{\alpha+1}{\alpha+1/2}}{2}$. It is easy to check that for $\alpha > 0$ $H \in (0.5, 1)$.

(v) The model is not a second-order stationary process due to the cascade based construction.

3.5 COMPARISON WITH OTHER MULTIFRACTAL MODELS

We first summarize the reasons which explain the suitability of the described multifractal model for network traffic modeling: (1) it is a positive process, hence reasonable for the simulation of the traffic counting processes; (2) it captures the full multifractal characteristics defined by the scaling function $\tau_0(q)$ and the moment factor $c(q)$; (3) it has approximately lognormal marginal distribution, which seems to match the real traffic; (4) it also has LRD correlation structure, which is an important property of high-speed LAN/WAN network traffic. The model in general is not stationary but it does not prevent from practical applications. For example, the model can be very well used for network traffic synthesis over small timescales ranged from a few hundreds of milliseconds downwards [8], where multiscaling and lognormal properties dominate.

Since the observation of the flexible scaling structure in some WAN traffic environments [3, 41, 5] network researchers have suggested several multifractal models for characterization of these traffic flows among which two district approaches can be identified. The first one uses multifractal time for subordinating a monofractal process (e.g. fBm) to model multifractals, e.g. in [42]. The disadvantages of this approach lie in the presence of some negative values and Gaussian marginal of the model synthetic processes, which is not suitable for network traffic simulation. The other approach is based on the multiplicative cascades

[7, 6, 29, 30, 31]. In general, multiplicative cascades are very attractive for traffic modeling. They are positive processes, easy to generate, and also possessing a plausible explanation for the origin of multiscaling properties in the traffic, see [6, 8] for more details. However, authors in [29] and [30] only fit the model to the LRD structure of the measured traffic, thus these models do not capture the multifractal characteristics of the traffic which is also ruled by the higher order moments. Cascade models in [7, 6] provide a better fit to the traffic multifractality but require N parameters for 2^N synthetic data. Then it is difficult to use these models in analytical approaches, e.g., in queueing performance estimations for multifractal traffic input.

In contrast, out traffic model suggests an alternative method of cascade modeling with only three parameters. The model provides the closed analytic form of the characteristic functions of multifractality with properties close to the real data traffic.

4 Queueing Analysis for Multifractal Traffic

A performance analysis of multifractal queues is discussed in this section. We first provide an approximation formula for the tail of the queue with multifractal traffic input. Then the practical use of the formula is presented.

4.1 Queueing model

We consider a simple queueing model: a single server queue in continuous time, the serving principle for offered work is defined to be FIFO (First In, First Out), the queue has infinite buffer and constant service rate s. Denote by $X(t)$ the total size of work arriving to the queue from time instant $-t$ in the past up to this moment, time instant 0. The so called *workload process* $W(t)$ is the total amount of work stored in the buffer in time interval $(-t, 0)$, i.e.,

$$W(t) = X(t) - st \tag{18}$$

Our interest, however, is the current buffer length of the queue, denoted by Q. This is the queue length in the equilibrium state of the queue when the system has been running for a long time and the initial queue length has no influence. If this state of the system does exist, i.e., stationarity and ergodicity of the workload process hold, and the stability condition for the system is also satisfied, i.e., $\limsup_t \mathbb{E}[X(t)]/t < s$, then:

$$Q = \sup_{t\geq 0} W(t), \tag{19}$$

where $W(0)$ is assumed to be 0. This equation is also referred to as *Lindley's equation*.

The input process $X(t)$ is taken to be a generally defined multifractal process (see Eq. 2) by Mandelbrot *et al.* in [23]. The definition presents multifractal processes in terms of moments, which leads to a more intuitive understanding of multifractality. We also assumed that the investigated TCP queueing occurs somewhere in the high-speed backbone network, where the feedback property of TCP traffic has negligible impact on the traffic aggregation.

4.2 Approximation for queue tail probabilities

We now state our main result, which is a first-order approximation:

Proposition 1 *The probabilities for the queue tail asymptotic of a single queueing model with general multifractal input is accurately approximated by:*

$$\log(\mathbf{P}[Q > b]) \approx \min_{q>0} \log \left\{ c(q) \frac{\left[\frac{b\,\tau_0(q)}{s(q-\tau_0(q))}\right]^{\tau_0(q)}}{\left[\frac{b\,q}{q-\tau_0(q)}\right]^{q}} \right\} \tag{20}$$

for large q, where $\tau_0(q) := \tau(q) + 1$. The scaling function $\tau(q)$ and $c(q)$ are the functions which define the multifractal input process.

Proof
Using Lindley's equation the tail probabilities of queue length can be rewritten of the form: $\mathbf{P}[Q > b] = \mathbf{P}[\sup_{t\geq 0} W(t) > b]$. First let consider the quantity $\mathbf{P}[W(t) > b]$:

Replacing $W(t)$ by Eq. 18 we have

$$\begin{aligned}
\mathbf{P}[W(t) > b] &= \mathbf{P}[X(t) - st > b] \\
&\leq \mathbf{P}[|X(t)| > b + st] \qquad (21)\\
&= \mathbf{P}[|X(t)|^q > (b + st)^q], \qquad \text{for } q > 0 \\
&\leq \frac{E[X(t)^q]}{(b + st)^q}. \qquad (22)
\end{aligned}$$

The last equality is applied using Markov's inequality.

Since the input process is multifractal defined by Eq. 2 then:

$$\begin{aligned}
\mathbf{P}[W(t) > b] &\leq \frac{c(q)t^{\tau_0(q)}}{(b + st)^q} \\
\sup_{t\geq 0} \mathbf{P}[W(t) > b] &\leq \sup_{t\geq 0} \frac{c(q)t^{\tau_0(q)}}{(b + st)^q} =: \sup_{t\geq 0} f(t). \qquad (23)
\end{aligned}$$

By straightforward calculation it is easy to see that that derivative of $f(t)$ equals zero when

$$t_0 = \frac{b\tau_0(q)}{s[q - \tau_0(q)]} > 0.$$

The second derivative of $f(t)$ at this point is

$$f''(t_0) = c(q)t_0^{\tau_0(q)-1}(b + st_0)^{-q-1}s[\tau_0(q) - q] < 0,$$

thus $f(t)$ has its maximal value at $t = t_0$. Note that we have assumed that $q > \tau_0(q)$, this is justified by the fact that in the monofractal case we have $\tau_0(q) = qH < q$ and by the concavity of $\tau(q)$.

Therefore

$$\sup_{t\geq 0} \mathbf{P}[W(t) > b] \leq \sup_{t\geq 0} f(t) = c(q) \frac{\left[\frac{b\,\tau_0(q)}{s(q-\tau_0(q))}\right]^{\tau_0(q)}}{\left[\frac{b\,q}{q-\tau_0(q)}\right]^{q}}$$

thus

$$\log\left(\sup_{t\geq 0} \mathbf{P}[W(t) > b]\right) \leq \log\left(c(q) \frac{\left[\frac{b\,\tau_0(q)}{s(q-\tau_0(q))}\right]^{\tau_0(q)}}{\left[\frac{b\,q}{q-\tau_0(q)}\right]^{q}}\right)$$

for arbitrary $q > 0$. Then

$$\log\left(\sup_{t\geq 0} \mathbf{P}[W(t) > b]\right) \leq$$

$$\leq \min_{q>0} \log\left(c(q) \frac{\left[\frac{b\,\tau_0(q)}{s(q-\tau_0(q))}\right]^{\tau_0(q)}}{\left[\frac{b\,q}{q-\tau_0(q)}\right]^{q}}\right). \quad (24)$$

For a large class of stochastic processes (including fBm) the following limit holds [43]:

$$\lim_{b\to\infty} \frac{\log(\mathbf{P}[Q > b])}{\log(\sup_{t\geq 0} \mathbf{P}[W(t) > b])} = 1. \quad (25)$$

In addition,

$$\log(\mathbf{P}[Q > b]) \geq \log(\sup_{t\geq 0} \mathbf{P}[W(t) > b]), \quad (26)$$

then the right-hand side of Eq. 24 is a upper bound of a lower bound on $\log(\mathbf{P}[Q > b])$. The used inequalities in Eq. 26 and Eq. 22 become tight for finite large b. Thus our first-order approximation for the queue tail asymptotics is the following:

$$\log(\mathbf{P}[Q > b]) \approx \min_{q>0} \log\left(c(q) \frac{\left[\frac{b\,\tau_0(q)}{s(q-\tau_0(q))}\right]^{\tau_0(q)}}{\left[\frac{b\,q}{q-\tau_0(q)}\right]^{q}}\right),$$

b large. □

For positive multifractal processes, i.e. $X(t) > 0$, Eq. 21 is an equality. In addition, the approximation in Eq. 26 and the inequality in Eq. 22 turn to be more accurate approximations as b tends to infinity. Thus the presented approximation is supposed to be asymptotically tight. The tightness and accuracy of the approximation is also experimentally investigated in Section V.

Considering the formula in Eq. 20 we see that it has an implicit form and just the given form of the characteristic functions $c(q)$ and $\tau(q)$ can provide the final result. The reason behind this is that the definition for the class of multifractal processes gives no restrictions on the functions $c(q)$ and $\tau(q)$ (beyond that $\tau(q)$ is concave). *Our conjecture is that the analysis of queueing systems with general multifractal input may produce some similar general results*. It means that there is no general queueing behavior for these systems as the Weibullian decay in the case of Gaussian self-similar processes [14]. An actual multifractal model will determine, for example, the queue length probabilities of the system.

4.3 Applications

4.3.1 Fractional Brownian motion

As a simple application first we consider a monofractal Gaussian process, called fractional Brownian motion (fBm). fBm is self-similar which is a simple case of monofractality and it is also Gaussian. The increment process of fBm is called fractional Gaussian noise (fGn). Queueing analysis of a single queue with fBm input is first presented by Norros [14] which showed the Weibullian decay for the asymptotic tail behavior, i.e., $\mathbf{P}[X > x] \sim \exp(-\gamma x^{\beta})$ with $\beta \leq 1$. This result is also justified by Large Deviation techniques in [35]. Applying this input process model to our formula should show its use and robustness when comparing to these available results.

First we prove that any Gaussian process with scaling property is in the class of monofractal processes. Furthermore we give the explicit forms for $\tau(q)$ and $c(q)$.

Consider the following proposition:

Proposition 2 *A Gaussian process with scaling property, i.e. fBm, is monofractal with parameters*

$$\begin{cases} \tau(q) &= \frac{q}{2}\left[\tau(2)+1\right]-1 \\ c(q) &= \frac{[2c(2)]^{q/2}}{\sqrt{\pi}}\Gamma\left(\frac{q+1}{2}\right), \end{cases}$$

where $\Gamma(\cdot)$ denotes the Gamma function,
$\Gamma(z) = \int_0^{+\infty} x^{z-1}\exp(-x)\mathrm{d}x,\ z > 0.$

The proof of this Proposition is provided in Appendix 7.1.

Turning back to our case of fBm with $c(2) = 1$ and $\tau(2) = 2H - 1$ where H is referred to as the Hurst parameter, we have

$$\begin{cases} \tau(q) &= qH-1 \\ c(q) &= \frac{2^{q/2}}{\sqrt{\pi}}\Gamma\left(\frac{q+1}{2}\right). \end{cases}$$

Inserting these two functions into our formula in Eq.20 we get

$$\begin{aligned} &\log(\mathbf{P}[Q > b]) \approx \\ &\approx \log\left(\min_{q>0}\left\{\frac{2^{q/2}}{\sqrt{\pi}}\Gamma\left(\frac{q+1}{2}\right)\frac{\left(\frac{bH}{s(1-H)}\right)^{qH}}{\left(\frac{b}{1-H}\right)^{q}}\right\}\right) \\ &=: \log(\min_{q>o} g(q)). \end{aligned} \quad (27)$$

The minimum value of the $g(q)$ for $q > 0$ function can be easily determined by taking its derivatives. The result is the following:

$$\log(\mathbf{P}[Q > b]) \approx \log(\min_{q>o} g(q))$$
$$= \log\left(\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\Psi^{-1}\left(\log K\right)\right)}{K^{\Psi^{-1}(\log K)-1/2}}\right)$$
$$=: \log(T_{fBm}(H, s, b)), \qquad (28)$$

where $K = K(H, s, b) =$ $= \frac{1}{2}b^{2(1-H)}s^{2H}(1 - H)^{-2(1-H)}H^{-2H}$, $\Psi(\cdot)$ is the *digamma* function, $\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and $\Psi^{-1}(\cdot)$ denotes the inverse function of $\Psi(\cdot)$.

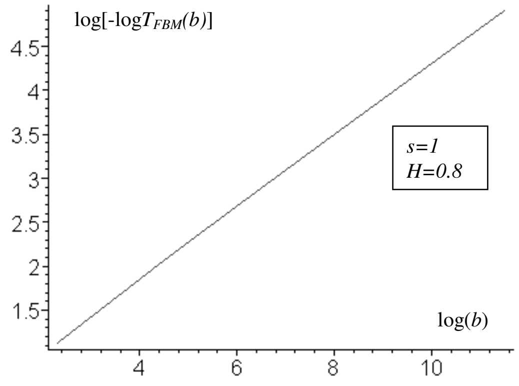

Figure 4: *By setting fixed values for H and s, the line in the log-log plot of $-\log T_{fBm}(b)$ versus b clearly shows the Weibullian decay for $T_{fBm}(H, s, b)$.*

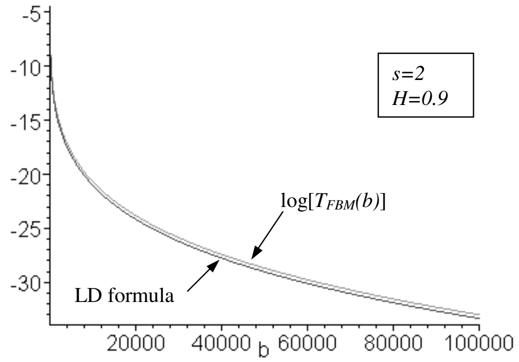

Figure 5: *Our approximation compared to the Large Deviation technique result: the two plots almost coincide for all values of the queue size.*

The $T_{fBm}(H, s, b)$ function is quite complex with the presence of Gamma, digamma, and its inverse function. However, we have quite a good approximation of $T_{fBm}(H, s, b)$:

Proposition 3 *The first-order approximation*

$$\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\Psi^{-1}\left(\log x\right)\right)}{x^{\Psi^{-1}(\log x)-1/2}} \approx \exp(-x) \qquad (29)$$

holds for large x, $x > 0$.

The proof and the precise sense of this approximation can be found in Appendix 7.2.

Applying this approximation we find that the queue tail for the fBm case satisfies:

$$\log\left(T_{fBm}(H, s, b)\right) \approx$$
$$\approx -\frac{1}{2}b^{2(1-H)}s^{2H}(1 - H)^{-2(1-H)}H^{-2H}, \quad b \text{ large.} \quad (30)$$

Eq. 30 shows the Weibullian decay of this queue which was first recognized and proven by Norros [14]. Numerical evaluations of the result are presented in Fig. 4 and Fig. 5. In Fig. 4 we fix the values of H and s and then calculate the values of the queue tail approximation $T_{fBm}(H, s, b)$ versus the queue size b and then plot it in the log-log scale. The linearity of the plot also demonstrates the Weibullian decay.

Now we compare our result to the result obtained by Duffield and O'Connell. The asymptotic formula for queue tail probabilities provided by Large Deviation technique presented in [35] is

$$\lim_{b\to\infty} b^{-2(1-H)} \log \mathbf{P}[Q > b] = -\inf_{c>0} c^{-2(1-H)} \frac{(c + s)^2}{2}$$
$$\Leftrightarrow \log \mathbf{P}[Q > b] \to$$
$$\to -\frac{1}{2}b^{2(1-H)}s^{2H}(1 - H)^{-2(1-H)}H^{-2H}, \text{ as } b \to \infty \quad (31)$$

where s also denotes the service rate. Therefore we can conclude that our approximation yields the Large Deviation result, see Eq. 30 and Eq. 31. The two results are depicted in Fig. 5 and we can see that the plots almost coincide for all calculated values of the queue size.

Our conclusions can be summarized in two main points: (i) the asymptotic tail approximation for the case of fBm has Weibullian decay; (ii) this result is also consistent with the formula presented by Norros [14] and by Duffield *et al.* with Large Deviation technique [35].

In the case of $H = 1/2$ (Brownian motion) the above formula results in $\log \mathbf{P}[Q > b] \approx -2sb/\sigma^2$ where σ^2 denotes the variance of the process, which is in agreement with the queueing formula known from the theory of Gaussian processes [44, 35].

4.3.2 Practical solutions

We show here the practical use of the formula. Assume that we are interested in the behavior of the tail of the steady-state buffer occupancy (queue length) distribution at a specific multiplexer in our network. The first step should be the fine resolution measurements of the input process.

We also assume that the input process exhibits multifractal scaling properties. Then the scaling function $\tau(q)$ and the moment factor $c(q)$ can be estimated from the collected data for some available parameters $q > 0$. *We emphasize the importance of the moment factor $c(q)$ as the quantity factor of multifractal processes which is sometimes neglected in a number of studies dealing with multiscaling properties of the high-speed network traffic. The scaling function $\tau(q)$ defines only the quality of multiscaling and it is not enough for the description of a multifractal model and therefore for the analysis of queueing models with multifractal input processes.*

Now we suggest two practical methods for the approximation of the queue tail distribution:

1. Given the service rate s and the two sets $\{c(q)\}$ and $\{\tau(q)\}$, using Eq. 20 the approximation of $\log(\mathbf{P}[Q > b])$ can be computed for each value of b. Note that we have no need to estimate $c(q)$ and $\tau_0(q)$ for all positive value of q, which is an impossible task. In fact, the largest value of q we should consider depends on the interested finite queue length of the involved queue length probability. This method is very simple but it is the more useful from network planning and capacity dimensioning point of view since we are only interested in some values of the tail probabilities. We mainly focus on the practical use of this method in this study.

2. The input process is fitted to a multifractal model. The two measured sets of $c(q)$ and $\tau(q)$ are fitted by $\tilde{c}(q)$ and $\tilde{\tau}(q)$. Then the analysis of the Eq. 20 with these functions can result in simple closed form of the queue tail probabilities. We use this method when studying the queue tail behavior of a multifractal model.

5 A Practical Case Study

In this section we examine the effectiveness of the presented multifractal model in the simulation of some real data traces. After fitting the model parameters, synthetic data is generated and compared with the real traces in a queueing performance analysis. We also show the applicability and validation of the presented queueing result on these traffic traces.

5.1 Analyzed data traffic

We present in this section our analysis for three measured network traffic, denoted by DATASET1, DATASET2, and DEC-PKT-2. The first two data traces were selected from our real traffic measurements carried out in busy hours of nearly two weeks in November 2000, from day 4 to day 16. The measurement point is located at a busy link connecting a set of the inner LANs of Informatics Building, Budapest University of Technology and Economics, with the outside world. The traffic traces, captured by `tcp-dump`, were the aggregated traffic of about 100 workstations used by staff member, PhD students, student laboratories, and they are the mixture of a number of application protocols such as HTTP, FTP, SSH, Telnet, and SMTP. During the measurement, more than 7 million IP packets were recorded. We select two data sets, DATASET1 and DATASET2, for analysis. The both sets contain 2^{17} data samples of IP traffic bytes, counted in 100 byte units for simple calculation, arriving in consecutive time intervals of 60ms. These two traces are found to exhibit multifractal properties.

In addition, as discussed, our queueing approximation can be also applied to both multifractal and monofractal traffic cases, so we also choose an known traffic trace (the DEC-PKT-2), which has uniscaling structure, available at the Internet Traffic Archive [45] for queueing analysis. The trace contains an hour's worth of all wide-area traffic each between Digital Equipment Corporation and the rest of the world (about 3 million IP packets). The DEC-PKT-2, which is the packet arrival count series of time sample of 3 milliseconds, is constructed from the raw data.

5.2 Analysis results

The multifractal analysis, model fitting, and queueing performance analysis for the DATASET1, DATASET2, and the DEC-PKT-2 are summarized in Fig. 6, Fig. 7, and Fig. 8, respectively. The details are presented below.

Figure 6(a) and (b) presents the multifractal analysis and model fitting results for the DATASET1. As seen in Fig. 6(a) the concavity of the estimated scaling function shows evidence for multiscaling structure of the DATASET1. We can also see in this figure the theoretical multiscaling function of the multiplicative cascade with parameter $\alpha_1 = 9.36$ of the Beta(α, α)-distributed multipliers. As it is seen, the multiscale function of the cascade process (also of the model) provide a very tight fit to the estimated curve. The next step is to determine the value of the parameters m and σ of the lognormal distribution to account for the difference between the logarithm of the estimated moment factor and the moment factor of the multiplicative cascade. The two moment factors and their difference are plotted in Fig. 6(b). We find that the lognormal distribution with $m_1 = 0.57$ and $\sigma_1 = 0.23$ is adequate for this goal. We present in Fig. 6(c) the multifractal spectrum of the original data sets comparing to of some synthetic data sets generated using our multifractal model. (Exactly, the half multifractal spectrums are seen in the figure, which correspond to the positive half line of the moment order q defined for multifractals.) It can be observed that the spectrums of the synthetic cases (dashed line) are nearly matched the original set (solid line).

Next we study the queueing performance analysis. We

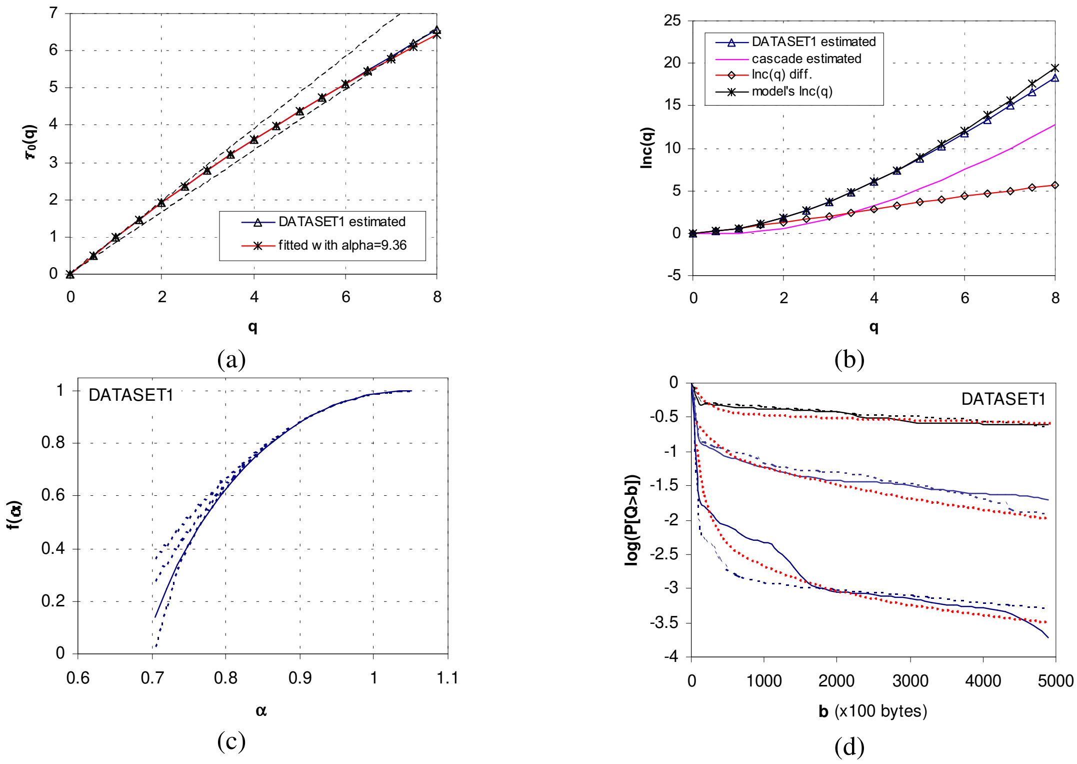

Figure 6: *Analysis results for the DATASET1: (a) the scaling function $\tau_0(q)$; (b) the moment factor $c(q)$; (c) multifractal spectrum of the data set (solid line) and of some data synthetics (dashed line) generated by our model; (d) queue tail probabilities at different server utilizations of* 0.9, 0.7, *and* 0.5 *(from top to bottom): DATASET1 (solid line), the corresponding synthetics (dashed line), and our approximation using the estimated $\tau_0(q)$ and $c(q)$ (dotted line).*

present in Fig. 6(d) the observed queue tail probabilities for the DATASET1 case. The analysis is considered at different server utilizations ρ. In the plot the solid lines present the tail probabilities of the real trace, the dashed lines the results of the trace simulation, and the dotted lines the theoretical values calculated applying our approximation form using the estimated multifractal characteristics functions. The curves, from top to bottom, correspond to $\rho = 0.5$, 0.7, and 0.9, respectively. The results show that the queueing behavior of the real trace is well captured by the model. In addition, we also observe that our queueing approximation can predict the tail probabilities accurately.

A similar analysis is carried out for the DATASET2 and the results are presented in Fig. 7. The multifractal analysis (seen in Fig. 7(a), (b), and (c)) also indicates the multiscaling structure for this data set. Our model can be fitted to the estimated multifractal characteristic functions with $\alpha_2 = 11.003$, $m_2 = 0.46$, and $\sigma_2 = 0.15$. As seen in the figure the synthetic data series generated by our model are found to fit the original sets in all respects of multifractal: the scaling functions, the multifractal spectrum, and the moment factor. Queueing analysis (Fig. 7(d)) is also justified the observations as experimented in the case of the DATASET1. The data set and the model synthetic series exhibit nearly the same queueing behavior in the investigated queueing system. The tail probabilities approximation by our queueing formula using the estimated scaling function and the moment factor is also found to be accurate.

Our analysis shows that the DEC-PKT-2 data set, however, has the exact monofractal structure and can be well modelled by statistical self-similarity with Hurst parameter $H = 0.8$ (see Fig. 8(a)). In this case the multifractal spectrum is only one point seen in Fig. 8(c) at (0.8,1), marked by a small square (○). Our queueing formula deals with the general multifractal inputs so it can also be applied in this case. The queueing analysis results are shown in Fig. 8(d). The queue tail simulation of the data series is plotted in the figure by solid lines, the approximation results are presented by the dotted lines. The curves, from top to bottom, belong to the case of server utilizations of 0.9 and 0.7. As seen in the figure the analysis also provides the correct queueing results in this case. These results (and our general experiences with several traces not presented in this paper) validate our queueing formula in practice cases.

6 Conclusions

In this paper we presented a comprehensive approach for multifractal performance analysis from the characterization and modeling of multifractal traffic to the queueing

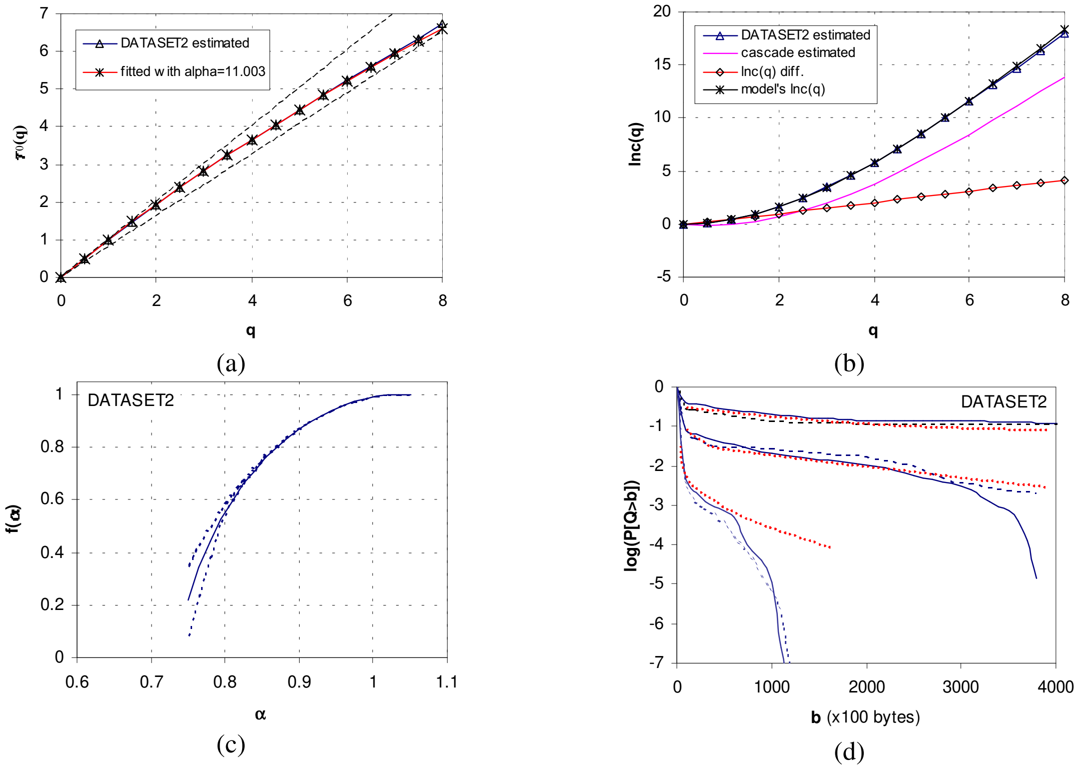

Figure 7: *Analysis results for the DATASET2: (a) the scaling function $\tau_0(q)$; (b) the moment factor $c(q)$; (c) multifractal spectrum of the data set (solid line) and of some data synthetics (dashed line) generated by our model; (d) queue tail probabilities at different server utilizations of* 0.9*,* 0.7*, and* 0.5 *(from top to bottom): DATASET2 (solid line), the corresponding synthetics (dashed line), and our approximation using the estimated $\tau_0(q)$ and $c(q)$ (dotted line).*

performance analysis. We introduced a new traffic model for multifractal network traffic based on the pairwise product of a multiplicative cascade and an independent, identically distributed lognormal process. The main statistical features and the LRD properties of the model were also presented. The practical applications of the model with measured traffic data were demonstrated and a queueing performance test has been carried out for validation.

We derived an asymptotic approximation of the steady-state queue length probabilities in a single server infinite capacity queue with constant service rate fed by a general multifractal input process. We showed that our queueing result gives the well-known Weibullian queue tail in case of the monofractal fractional Brownian motion input process. It was proven that the class of Gaussian processes with scaling properties is limited to monofractal processes and the characteristic functions were derived. Some impacts of multifractality on the queueing performance were also investigated and presented. Our queueing formula was applied for the case of Gaussian input processes and it verifies the results obtained from the theory of Gaussian processes. We demonstrated the practical applicability of our queueing approximation and validated the method by simulation. It was shown that our formula gives the correct result in the analysis of both multifractal and monofractal network traffic cases.

There are several interesting topics for further research. One of our goals is to apply for several LAN/WAN network environments and traffic types our multifractal performance analysis framework.

7 Appendices

7.1 Proof of Proposition 1

Denote by $X(t)$ the Gaussian process. Since $X(t)$ has scaling property it satisfies the general definition for multifractal process, i.e., $\mathbb{E}[|X(t)|^q] = c(q)t^{\tau(q)+1}$. Thus the variance of the $X(t)$ process should be $\sigma_t^2 = c(2)t^{\tau(2)+1}$. The Gaussian process $X(t) \sim N\left(0, c(2)t^{\tau(2)+1}\right)$ has the normal distribution and we have

$$f(x) = \frac{1}{\sqrt{2\pi c(2)t^{\tau(2)+1}}} \exp\left(-\frac{x^2}{2c(2)t^{\tau(2)+1}}\right).$$

The q^{th} moment of $X(t)$ can be calculated by the definition:

$$\begin{aligned}\mathbf{E}[|X(t)|^q] &= \int_{-\infty}^{+\infty} |x|^q f(x) \mathrm{d}x \\ &= 2\int_0^{+\infty} x^q \frac{1}{\sqrt{2\pi c(2)t^{\tau(2)+1}}}.\end{aligned}$$

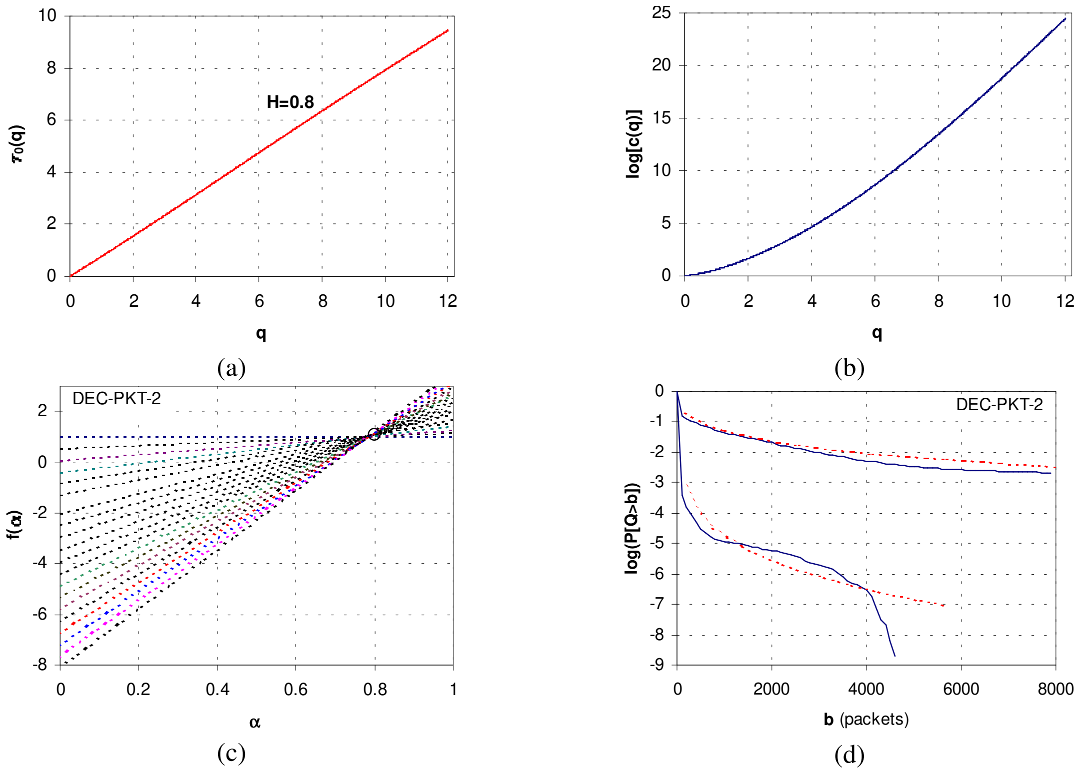

Figure 8: *Analysis results for the DEC-PKT-2: (a) the scaling function* $\tau_0(q)$*; (b) the moment factor* $c(q)$*; (c) single-valued multifractal spectrum of the data set* $H = 0.8$ *(marked by* $\circ$*); (d) queue tail probabilities at different server utilizations of* 0.9 *and* 0.7 *(from top to bottom): DEC-PKT-2 (solid line), our approximation using the estimated* $\tau_0(q)$ *and* $c(q)$ *(dotted line).*

$$. \exp\left(-\frac{x^2}{2c(2)t^{\tau(2)+1}}\right) \mathrm{d}x.$$

Introduce $y := \frac{x^2}{2c(2)t^{\tau(2)+1}}$. The formula above can be rewritten as follows:

$$\begin{aligned}\mathbf{E}[|X(t)|^q] &= \frac{2^{q/2}}{\sqrt{\pi}}\left[c(2)t^{\tau(2)+1}\right]^{q/2}. \\ &\quad . \int_0^{+\infty} y^{\frac{q-1}{2}} \exp(-y)\mathrm{d}y \\ &= \frac{[2c(2)]^{q/2}}{\sqrt{\pi}}\Gamma\left(\frac{q+1}{2}\right) t^{\frac{q}{2}[\tau(2)+1]}, \qquad (32)\end{aligned}$$

which concludes our proof. $\square$

7.2 Proof of Proposition 2

The approximation in Proposition 2 states that

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\frac{\Gamma\left[\Psi^{-1}(\log x)\right]}{x^{\Psi^{-1}(\log x)-1/2}} \approx \exp(-x), \qquad x \text{ large} \\ &\Leftrightarrow \log\Gamma\left[\Psi^{-1}(\log x)\right] \approx \\ &\qquad \left[\Psi^{-1}(\log x) - 1/2\right]\log x - x + \log\sqrt{\pi}, \qquad (33)\end{aligned}$$

By the *Stirling*'s formula we have

$$\log\Gamma(z+1) - (z+1/2)\log z + z \to \log\sqrt{2\pi} \quad \text{as } z \to \infty. \qquad (34)$$

Put $z+1 = \Psi^{-1}(\log x)$, hence $x = \exp\Psi(z+1) \to \infty$ as $z \to \infty$ by the monotonicity of $\Psi(\cdot)$. We obtain

$$\begin{aligned}&\log\Gamma\left[\Psi^{-1}(\log x)\right] - \left[\Psi^{-1}(\log x) - 1/2\right]. \\ &. \log\left[\Psi^{-1}(\log x) - 1\right] + \left[\Psi^{-1}(\log x) - 1\right] \to \\ &\qquad \to \log\sqrt{2\pi}, \quad \text{as } x \to \infty. \qquad (35)\end{aligned}$$

In [46] it is derived that

$$\lim_{z\to\infty}\frac{\Psi(z+1)}{\log z} = 1, \qquad (36)$$

hence $x = \Psi^{-1}\left[\omega(x)\log x\right] - 1$ where $\lim_{x\to\infty}\omega(x) = 1$. Using the approximation $\Psi^{-1}(\log x) - 1 \approx x$ for large x, we get the equation

$$\begin{aligned}&\log\Gamma\left[\Psi^{-1}(\log x)\right] - \left[\Psi^{-1}(\log x) - 1/2\right]\log x + \\ &\qquad + x \approx \log\sqrt{2\pi}. \qquad (37)\end{aligned}$$

The approximation above is also verified by numerical computations and it is shown that the approximation can be effectively applied for large x.

Then the approximation in Eq. 33 follows if we change the constant $\log\sqrt{2\pi}$ by $\log\sqrt{\pi}$. $\square$

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