# Cell Delay Variation in an ATM Multiplexer

Sándor Molnár[1] Søren Blaabjerg[2]

**Abstract**

In this paper Cell Delay Variation (CDV) in an ATM multiplexer is investigated in the case where a Constant Bit Rate (CBR) traffic stream is mixed with a background traffic stream that arrives in the form of independent and identically distributed batches with general batch size distribution (Batch Bernoulli process). An exact Markovian solution approach is presented which gives the entries in the underlying transition matrix in closed form. In addition an approximate diffusion approach that is both computational simple and provides insight into the point process properties of the CBR stream after passing through the multiplexer is presented. It is demonstrated that the diffusion approximation has nice properties: it is second order exact in the heavy traffic case, and it provides fast and reasonable accuracy results over a wide range of system parameters.

## 1 Introduction

For constant bitrate (CBR) services like voice and circuit emulation an important Quality of Service (QoS) parameter is the Cell Delay Variation (CDV). CDV covers the phenomenon that different cells of the same connection experience different delays in the switches due to varying traffic conditions. Since CBR services are expected to contribute a significant part of the traffic in the first applications of ATM, a successful evolution towards an ATM based B-ISDN is likely to depend, by many other factors, also on the ability to control the CDV.

A thorough understanding of how CDV arises, how it is affected by traffic from other sources, by its own bitrate, by the number switch buffers and multiplexing stages passed etc., is of great importance concerning setting of the Usage Parameter Control (UPC) parameters, concerning conformance testing, concerning bandwidth allocation and concerning the dimensioning of the playout buffer at the receiving end of the connection.

In the past few years many investigations have contributed towards an understanding of CDV. In [6] a thorough investigation of the $D + GI^{[X]}/D/1$ queue, based on generating functions and the theory of holomorfic functions, has been carried out with the main objective of finding the steady state waiting time at any time including the waiting time seen by the

[1]Department of Telecommunications & Telematics, Technical University of Budapest,
Sztoczek u. 2., H-1111 Budapest, Hungary
(e-mail: molnar@bme-tel.ttt.bme.hu)

[2]Department of Telecommunications, Technical University of Denmark,
DK-2800 Lyngby, Denmark
(e-mail: sbb@tele.dtu.dk)

CBR arrivals. In [11] another more direct Markovian approach has been applied and the emphasis was on the study of the point process properties of the departure process of the CBR stream. The results made an accurate setting of the UPC parameters possible. By modelling the queue length behaviour between CBR arrivals by a Brownian motion a computationally much simpler model was derived in [3]. Closed form expressions of the most important point process characteristics has been obtained. This diffusion model was also applied to analyze the superposition of CDV affected CBR cell streams [20]. The UPC issue has played a dominating role in the literature and a number of approaches exits to obtain the two peak cell rate parameters, CDV tolerance and peak emission interval, see [10], [12], [2] and [16]. The first results concerning the behaviour of CDV in a networking environment can be found in [4] and [19], where the results in [19] are based on the results of the same authors [18] for the single multiplexer case. For a recent introduction to CDV in general and to teletraffic aspects of CDV in particular, we refer to [13]. The CDV issue is also investigated by the standardization bodies [14] and ATM Forum [1].

In this paper we investigate the point process properties of the departure process of the CBR stream from the multiplexer, under the batch Bernoulli background traffic. As shown in [11] and [5] then the waiting time which the CBR cells experience in the ATM multiplexer constitutes a Markov process under this assumption. Our contribution consists of the following main components:

1. We extend the Markov model of Guillemin and Roberts [11] from Poissonian background traffic to batch Bernoulli traffic with arbitrary batch size distribution $GI^{[X]}$, and thereby with arbitrary peakedness. This is carried out by deriving analytical expressions for the entries of the transition matrix of the underlying Markov process.

2. We further develop the diffusion approach of Blaabjerg [3] for background traffic other than Poisson, to provide a computationally more feasible approximation. This is supported in two ways: (1) we evaluate the accuracy of the diffusion model in the heavy traffic limit where the multiplexer load approaches 1, and prove that the diffusion model in this limit is second moment exact, independently of the rate of the CBR stream; (2) we also provide numerical evaluation of the diffusion model for background traffic other than Poisson.

In the present paper we focus our attention on the core model of a single queue, periodic main traffic and uncorrelated background traffic, since this is the fundamental case that contains the most essential part of the analysis of e.g. [18] and [19]. On the other hand, we widen our scope by considering the *whole* point process of the CDV affected CBR stream, as opposed to focusing on a single inter-departure time interval.

In section 2 a brief summary of the Markov model of [11] is given. In section 3 the expressions for the entries in the transition matrix of the underlying Markov model are given. The diffusion model of [3] which yields closed form expressions for the distribution of the interdeparture time of CBR cells, the distribution for the number of CBR cells in a window, and Index of Dispersions for Intervals (IDI), is briefly outlined, and the limit behaviour when the load on the $D + GI^{[X]}/D/1$ queue approaches 1, is characterised. In section 4 a numerical investigation of the diffusion model in a number of different scenarios is carried out.

# 2 Characterization of the CDV affected stream by a Markov model

In this section the characterization of the departure process of the CBR cell stream after the output of the multiplexer is outlined (see [11] for further details).

Consider a CBR cell stream with interarrival time $T$ ($T$ being integer). Choose the time such that cell $n$ is transmitted at time $nT$ from the source and arrives immediately to the multiplexer. Let $W_n$ denote the waiting time of cell $n$ that is the time from arrival at the multiplexer until start of service.

Two assumption are made in the model:

- The sequence $W_n$ is stationary with distribution $w_k = P\{W_i = k\}$.
- The dependence between waiting times of successive cells is assumed first order Markovian, and characterized by the transition matrix $\mathbf{Q}$ in which entry $(j,k)$ is:

$$q_{j,k} = P\{W_i = k \mid W_{i-1} = j\} \quad j,k \geq 0 \tag{1}$$

Define $\tau_n = nT + W_n + 1$. Thus $\tau_n$ denotes the departure time of cell no $n$. Define the *shifted interdeparture time* of cell $n$ as: $U_n = \tau_n - nT - \tau_0$ (see Figure 1). It is a fundamental random variable from which all CDV characteristics of interest can be derived. Let $f_n(k) = P\{U_n = k\}$ be the probability distribution of $U_n$ for cell $n$. It can be seen [11] that

$$f_n(k) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} w_i q_{i,i+k}^{(n)} & \text{if } k \geq -nT + 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $q_{j,k}^{(n)}$ denotes entry $(j,k)$ in the $n$'th power of the transition matrix $\mathbf{Q} = \{q_{j,k}\}$.

Next we show how the usual point process characteristics can be derived from $U_n$. We first investigate the *interval representation.*

### Interdeparture time distributions

The interdeparture time distribution between cell 0 and $n$ is simply obtained by the translation $nT$ of the shifted interdeparture time distribution $U_n$ i.e.

$$P\{\tau_n - \tau_0 = k\} = P\{U_n = k - nT\} = f_n(k - nT) \tag{3}$$

**Index of dispersions for intervals**

The index of dispersion can be obtained from the definition as:

$$c_n^2 = \frac{nVar(\tau_n - \tau_0)}{E^2(\tau_n - \tau_0)} = \frac{Var(U_n)}{nT^2} = \frac{\sum_{k=-nT+1}^{\infty} k^2 f_n(k)}{nT^2} \tag{4}$$

since $E(U_n) = 0$.

For the *counting representation* the number of counts is fundamental and is measured in two ways:

**Number of departures in a window starting just after a departure**

Let $N(\tau_j, \tau_j + t)$ denote the number of departures in a window of length $t$ starting just after an arbitrary CBR cell departure $\tau_j$. The probability distribution is

$$\begin{aligned} P\{N(\tau_j, \tau_j + t) = n\} &= P\{U_n + nT \le t\} - P\{U_{n+1} + (n+1)T \le t\} \\ &= \sum_{j \le -nT+t} f_n(j) - \sum_{j \le -(n+1)T+t} f_{n+1}(j). \end{aligned} \tag{5}$$

**Number of departures in an arbitrary window**

Based on a basic result in point process theory (see e.g. section 4.2 in [7]) a simpler formula for the distribution of the number of departures in an arbitrary window than the one presented in [11] can be obtained. Let $t_0$ denote an arbitrary point in time and $N(t_0, t_0 + t)$ denote the number of departures in $]t_0, t_0 + t]$. Then

$$P\{N(t_0, t_0+t) \ge n\} = P\{Y + X_2 + \ldots + X_n \le t\} = \frac{1}{T} \sum_{u=1}^{t} \sum_{k=1}^{u} (f_{n-1}(k - [n-1]T) - f_n(k - nT)) \tag{6}$$

in which $Y$ denote the forward recurrence time, $X_i$ denote the interdeparture time between cell $i-1$ and $i$. From (6) we obtain

$$P\{N(t_0, t_0+t) = n\} = \frac{1}{T} \sum_{u=1}^{t} \sum_{k=1}^{u} f_{n-1}(k-(n-1)T) - 2f_n(k - nT) + f_{n+1}(k-(n+1)T) \tag{7}$$

**Limit Distributions**

The Markovian assumption implies that in the limit $n \to \infty$, $q_{j,k}^{(n)} \to w_k$ independent of $j$

yielding the limit distribution

$$f_\infty(k) = \sum_{i=0}^{\infty} w_i w_{i+k} \tag{8}$$

and

$$P\{N(\tau_j, \tau_j + t) = n\} = \sum_{j=-(n+1)T+t+1}^{-nT+t} f_\infty(j) \tag{9}$$

# 3 CDV due to the FIFO queue

Consider a single server infinite capacity discrete time FIFO queue receiving the superposition of a CBR stream with interarrival time $T$ and an interfering background stream which arrives to the queue in the form of independent and identically distributed batches with a general batch size distribution denoted by $b(k)$ and generating function $B(z) = \sum_{i=0}^{\infty} b(k) z^k$ (Figure 2). With these assumptions the load $\rho$ on the queue is

$$\rho = B'(z) + 1/T = E(B) + 1/T \tag{10}$$

where $E(B) = B'(z)$ is the mean number of background arrivals in a slot.

## 3.1 An exact solution approach

The cells arriving from the background cell stream in the same time slot are served in random order, but when there is also a cell arriving from the CBR stream it is served first. This serving rule will give a tight lower bound on the amount of CDV which the background stream puts on the CBR cell stream [18]. With the assumption that all actions in the queue occur at the time slot boundaries, the queue length seen by an arriving CBR cell is a Markov process and it equals the waiting time of the CBR cell.

The transition matrix $Q = \{q_{j,k}\}$ can be computed by

$$q_{j,k} = Q(j, k-1) - Q(j, k) \tag{11}$$

where

$$Q(j, k) = P\{W_i > k \mid W_{i-1} = j\} = \sum_{n \geq 0} P_n(j, k) \tag{12}$$

with $P_n(j, k) = P\{W_i > k \mid W_{i-1} = j$ and $n$ arrivals in $]Ti - T, Ti]$ $\} b^{T*}(n)$, where $b^{T*}(n)$ is the $T$-fold convolution of the batch size distribution, which evaluated at $n$ represents the probability of $n$ arrivals in $]Ti-T, Ti]$. It can be shown (see Appendix A) that

$$P_n(j,k) = \begin{cases} 0 & j+1 \geq T \text{ and } n \leq T+k-j-1 \text{ or} \\ & \quad j+1 < T \text{ and } k \geq n \\ b^{T*}(n) & n > T+k-j-1 \\ \sum_{s=1}^{n-k} b^{s*}(k+s) \times & \\ \qquad \times b^{(T-s)*}(n-k-s)\frac{T-n+k}{T-s} & j+1 < T \text{ and } k < n \leq T+k-j-1 \end{cases} \tag{13}$$

The stationary waiting time distribution can be found from solving the equilibrium system $\mathbf{w} = \mathbf{wQ}$ where $\mathbf{w} = (w_0, w_1, ..., w_i, ...)$ and $\mathbf{Q} = \{q_{j,k}\}$. In numerical applications, a truncation of the state space is in general necessary.

[5] provides an appealing alternative approach by giving a closed form expression for the generating function of the steady state queue length distribution. Moreover, [5] also gives the generating function of the waiting time for cell $n+1$, conditioning on the event that the waiting time for cell $n$ is $i$. These results give a complete characterization, as they enable the computation of the entries of the transition matrix. On the other hand, they involve the additional computational burden of matrix inversion and the determination of $T-1$ boundary probabilities. In the present paper, we have chosen the direct approach to show that the transition matrix entries can also be found directly, without further essential side-computations. The two approaches both have advantages, the choice between them depends on the aspects of the particular application.

## 3.2 Diffusion approximation

The approach presented above is attractive because it is exact and the fundamental solution of the transition matrix is given in closed form. However, the computational burden is high and the numerical complexity increases without bound when the load of the multiplexer approaches one. Therefore an accurate approximation is needed which maintains the important qualities of the model but for which the fundamental point process quantities can be easily computed. As a candidate we suggest the diffusion approximation which fulfills these requirements.

### 3.2.1 The model

Our diffusion model is based on the idea that the evolution of the queue length (or virtual waiting time) between CBR arrivals is approximated by a reflected Brownian motion. Markovian dependencies are assumed between successive CBR cell waiting times in the queue as in the former section.

Let $\tilde{W}_t$ denote the waiting time a fictitious observer would experience if he joined the diffusion queue at time $t$ (the virtual waiting time at time $t$). The probability of $\tilde{W}_t \leq x$ conditioned on $\tilde{W}_0 = y$ is, for a Brownian motion with drift $m$ ($m$ assumed smaller than zero in order to ensure a stable queue), variance $\sigma^2$ and a reflection in zero, in section 2.8 of

[17] derived to be:

$$P\{\tilde{W}_t \leq x \mid \tilde{W}_0 = y\} \quad = \quad \begin{cases} \Phi(\frac{x-y-mt}{\sigma\sqrt{t}}) - e^{2\frac{mx}{\sigma^2}}\Phi(\frac{-x-y-mt}{\sigma\sqrt{t}}), & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{14}$$

where $\Phi$ denotes the standard Gaussian probability distribution. The right hand side of formula (14) is a distribution function in $x$ for all $y \geq 0$ and all $t > 0$, and it converges towards the exponential distribution with mean $\frac{-\sigma^2}{2m}$, independent of initial condition $y$, when $t$ tends to infinity.

The diffusion approximation is used for modeling the arrivals of cells belonging to both the CBR stream and the background stream, i.e. the queue length is not increased by one at each CBR cell arrival. These arrivals as well as the batch arrivals and the departures are taken into account by a proper choice of drift and variance in the diffusion process as shown in the next subsection.

The waiting time of CBR cell $n$, is approximated by:

$$W_n = \tilde{W}_{nT} \tag{15}$$

that is *the waiting time that CBR cell n experience in the queue is approximated by the virtual waiting time in the diffusion queue at time nT.*

A disadvantage with formula (14) is that there is a positive probability that the waiting time between time $t$ and time $t+u$ decreases more than $u$ which of course in the original $GI^{[x]} + D/D/1$ queue is impossible. If $t$ is the time of CBR arrival $n$ and $u = T$ then this would imply that CBR cell $n+1$ departs the queue before cell $n$. This weakness is inherent to the model and the model should only be used for values of $T$ for which this probability is low, implying that $T$ cannot be chosen too small.

### 3.2.2 Drift and variance computation

As in the diffusion model for the M/G/1 queue (see section 2.8 in [17]) the drift is $m = \rho - 1$ where $\rho$ denotes the load of the queue. The key idea of choosing the most appropriate variance is that *we match the decay rates of the stationary waiting time distribution in the diffusion approximation with the asymptotic decay rate of the* $D + GI^{[x]}/D/1$ *queue.*

It is known from [6] that the queue length seen by an arriving CBR cell is asymptotically geometric i.e. $P(Q > r) \approx (1/z_\infty)^r$, where $z_\infty$ is the dominant root of the corresponding $z$-transform which is found by solving the equation

$$B(z)^T - z^{T-1} = 0. \tag{16}$$

where $B(z)$ is the z-transform of the batch size distribution of the background process. The root $z_\infty$ is the root with smallest module outside the unit disk, and it is unique and real (see [6] in Appendix A3).

Since the diffusion decay rate is $\frac{-2m}{\sigma^2}$ we get:

$$\sigma^2 = -\frac{2m}{\ln(z_\infty)}. \tag{17}$$

### 3.2.3 The distribution of the interdeparture time

The shifted interdeparture time in the diffusion context is: $\tilde{U}_t = \tilde{W}_t - \tilde{W}_0$ (think of $t = nT$), and

$$\begin{aligned} P\{\tilde{U}_t \le x\} &= \int_0^\infty P\{\tau_t - \tau_0 - t \le x \mid \tau_0 = y\} dP\{\tau_0 \le y\} \\ &= \int_0^\infty P\{\tilde{W}_t \le x + y \mid \tilde{W}_0 = y\} dP\{\tilde{W}_0 \le y\} \qquad (18) \end{aligned}$$

The computation carried out in [3] shows that the time dependent shifted interdeparture time distribution function is given as:

$$\tilde{F}_t(x) = P\{\tilde{U}_t \le x\} = \begin{cases} \frac{1}{2} + \frac{1}{2}\Phi(\frac{x-mt}{\sigma\sqrt{t}}) - \frac{1}{2}e^{\frac{2mx}{\sigma^2}}\Phi(-\frac{x+mt}{\sigma\sqrt{t}}), & \text{for } x \ge 0 \\ \frac{1}{2}e^{-\frac{2mx}{\sigma^2}}\Phi(\frac{x-mt}{\sigma\sqrt{t}}) + \frac{1}{2}\Phi(\frac{x+mt}{\sigma\sqrt{t}}), & \text{for } x < 0 \end{cases} \qquad (19)$$

From this expression it is seen that the probability distribution function $\tilde{F}_t$ for $\tilde{U}_t$ fulfills the relation: $\tilde{F}_t(x) + \tilde{F}_t(-x) = 1$, thus implying that the density function for $\tilde{U}_t$ is symmetric.

As in the exact case a limit distribution emerges as $t \to \infty$ From (19) it is easily seen that

$$\tilde{F}_\infty(x) = P\{\tilde{U}_\infty \le x\} = \begin{cases} 1 - \frac{1}{2}e^{\frac{2mx}{\sigma^2}}, & \text{for } x \ge 0 \\ \frac{1}{2}e^{-\frac{2mx}{\sigma^2}}, & \text{for } x < 0 \end{cases} \qquad (20)$$

that is in the limit $t \to \infty$ the shifted interdeparture time is a two sided exponential distribution with rate $-\frac{2m}{\sigma^2}$ and $\frac{2m}{\sigma^2}$ respectively.

The interdeparture time distribution between cell $k$ and $k+n$ is obtained from the shifted interdeparture time distribution as

$$F_n(x) = P\{\tau_{k+n} - \tau_k \le x\} = P\{\tau_n - \tau_0 \le x\} = P\{\tilde{U}_{nT} \le x - nT\} = \tilde{F}_{nT}(x - nT) \qquad (21)$$

### 3.2.4 Index of dispersions for intervals

For IDI computation the variance of $\tilde{U}_t$ is needed, and in [3] it is found to be:

$$\begin{aligned} Var(\tilde{U}_t) &= \frac{\sigma^4}{2m^2}(2\Phi(-\frac{m}{\sigma}\sqrt{t}) - 1) + (2\sigma^2 t + m^2t^2)\Phi(\frac{m}{\sigma}\sqrt{t}) \\ &+ (\frac{\sigma^3}{m}t^{1/2} + m\sigma t^{3/2})\varphi(\frac{m}{\sigma}\sqrt{t}) \qquad (22) \end{aligned}$$

where $\varphi$ is the standard normal density function. By applying the expansion $\Phi(-x) = \frac{\varphi(x)}{x}(1 - \frac{1}{x^2} + O(x^{-4}))$, for $x \to \infty$, see e.g. problem 7.7.1 in [9] and evaluating $Var(\tilde{U}_t)$ at $t = nT$ we arrive at:

$$c_n^2 = \frac{\sigma^4}{2m^2T^2}\frac{1}{n} + O(n^{-3/2}e^{-\frac{m^2}{\sigma^2}nT}), \quad n \text{ large} \qquad (23)$$

which clearly shows that the relative variance of the departure process decreases with increasing time scales, i.e. the process is less bursty than a renewal process.

### 3.2.5 Number of departures in a window starting just after a departure

Consider an interval of the form $]\tau_j, \tau_j + t]$ starting just after the arrival of cell $j$. Then
$P\{N(\tau_j, \tau_j + t) \geq n\} = P\{\tilde{U}_{nT} + nT \leq t\} = \tilde{F}_{nT}(t - nT) =$

$$\begin{cases} \frac{1}{2} + \frac{1}{2}\Phi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1+m)}{\sigma}\sqrt{nT}) - \frac{1}{2}e^{\frac{2m}{\sigma^2}(t-nT)}\Phi(-\frac{t}{\sigma\sqrt{nT}} + \frac{(1-m)}{\sigma}\sqrt{nT}), \text{ for } t \geq nT \\ \frac{1}{2}e^{-\frac{2m}{\sigma^2}(t-nT)}\Phi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1+m)}{\sigma}\sqrt{nT}) + \frac{1}{2}\Phi((\frac{t}{\sigma\sqrt{nT}} - \frac{(1-m)}{\sigma}\sqrt{nT}), \text{ for } t < nT \end{cases} \tag{24}$$

### 3.2.6 The number of departures in an arbitrary window

Let $t_0$ denote an arbitrary point in time and consider the interval $]t_0, t_0 + t]$. Similarly as in (6) we can write

$$P\{N(t_0, t_0 + t) \geq n\} = P\{Y + X_2 + .. + X_n \leq t\} = \frac{1}{T}\int_0^t (F_{n-1}(u) - F_n(u))du \tag{25}$$

in which $Y$ denote the forward recurrence time, $X_i$ denote the interdeparture time between cell $i-1$ and $i$, and $F_i(u)$ denotes the distribution function of $\sum_{j=1}^{i} X_j$. Since $F_n(u) = P\{\tilde{U}_{nT} \leq u - nT\} = \tilde{F}_{nT}(u - nT)$ is given in closed form in formula (24) we may obtain the probability distribution of the number of departures in an arbitrary window by integrating (24) with respect to $t$. A complication arises due to the non-zero probability of cell $n+k$ departing the queue before cell $n$. If cell $n$ is the first departure in the window it may happen that cell $n+k$ is left of the window. However, if cell $n+p$ is the first departure after the end of the window it may happen that cell $n+p+k$ is inside the window. The easiest way to approximately capture this is by extending the integration in (25) from 0 down to $-\infty$ i.e. $P\{N(t_0, t_0 + t) \geq n\} \approx \frac{1}{T}\int_{-\infty}^{t}(F_{n-1}(u) - F_n(u))du$. The result of the integration then is:

$$P\{N(t_0, t_0 + t \geq n\} = G_{n-1}(t) - G_n(t) \tag{26}$$

in which $G_n(t)$ for $t < nT$ is given as:

$$\begin{aligned} G_n(t) = \quad & \frac{\sigma^2}{(-4m)}\frac{1}{T}\{e^{-\frac{2m}{\sigma^2}(t-nT)}\Phi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1+m)}{\sigma}\sqrt{nT}) \\ - \quad & \Phi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1-m)}{\sigma}\sqrt{nT})\} \\ + \quad & \frac{1}{2T}\{(t - (1-m)nT)\Phi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1-m)}{\sigma}\sqrt{nT}) \\ + \quad & \sigma\sqrt{nT}\varphi(\frac{t}{\sigma\sqrt{nT}} - \frac{(1-m)}{\sigma}\sqrt{nT})\} \end{aligned} \tag{27}$$

and for $t > nT$, $G_n(t)$ is given as:

$$G_n(t) = \quad \frac{\sigma^2}{(-4m)}\frac{1}{T}\{e^{\frac{2m}{\sigma^2}(t-nT)}\Phi(\frac{-t}{\sigma\sqrt{nT}} + \frac{(1-m)}{\sigma}\sqrt{nT})$$

$$
\begin{aligned}
- \quad & \Phi(\frac{-t}{\sigma\sqrt{nT}} + \frac{(1+m)}{\sigma}\sqrt{nT})\} \\
+ \quad & \frac{1}{2T}\{((1+m)nT - t)\Phi(\frac{-t}{\sigma\sqrt{nT}} + \frac{(1+m)}{\sigma}\sqrt{nT}) \\
+ \quad & \sigma\sqrt{nT}\varphi(\frac{-t}{\sigma\sqrt{nT}} + \frac{(1+m)}{\sigma}\sqrt{nT})\} + \frac{t-nT}{T} \qquad (28)
\end{aligned}
$$

where we have utilized the symmetry relation $\tilde{F}_t(x) + \tilde{F}_t(-x) = 1$ to obtain (28).

## 3.3 When the multiplexer load approaches 1

When the load on the multiplexer is very close to 1, the analysis simplifies, since the event of an empty system can be neglected. The first immediate consequence is that the departure process from the multiplexer constitutes a renewal process. We have

$$
\tau_n - \tau_{n-1} = W_n - W_{n-1} + T = 1 + N((n-1)T, nT) \qquad (29)
$$

Since $N((n-1)T, nT)$ is one CBR arrival + an independent sum of $T$ batches the interdeparture time has the distribution $1 + B^{T*}$ with variance $\sigma_B^2 T$, where $\sigma_B^2$ denotes the variance of the batch size distribution. The assumption of independence of batch arrivals from slot to slot then gives the renewal property. The index of dispersions for intervals (IDI) then becomes a horizontal line of value $\frac{\sigma_B^2}{T}$. The renewal result was also noted in [15] and [19].

The formulas in the diffusion approximation also simplifies in the limit $\rho \to 1$ $(m \to 0)$. From (19) we get that the shifted interdeparture time distribution between cell 0 and cell $n$ is

$$
\lim_{m \to 0} \tilde{F}_{nT}(x) = \Phi(\frac{x}{\sigma\sqrt{nT}}) \qquad (30)
$$

i.e. a normal distribution with zero mean and variance $\sigma^2 nT$.

The fact that the departure process becomes a renewal process in the limit $\rho \to 1$ $(m \to 0)$ is reflected in the IDI the following way. From (22) we get

$$
\begin{aligned}
Var(\tilde{U}_t) \quad = \quad & \frac{\sigma^4}{m^2}(-\frac{m}{\sigma}\sqrt{t}\varphi(0) + O(\frac{m^3}{\sigma^3}t^{3/2})) + 2\sigma^2 t\Phi(\frac{m}{\sigma}\sqrt{t}) \\
+ \quad & m^2t^2\Phi(\frac{m}{\sigma}\sqrt{t}) + \frac{\sigma^3}{m}t^{1/2}\varphi(\frac{m}{\sigma}\sqrt{t}) + m\sigma t^{3/2}\varphi(\frac{m}{\sigma}\sqrt{t}) \\
= \quad & \sigma^2 t\frac{\varphi(\frac{m}{\sigma}\sqrt{t}) - \varphi(0)}{\frac{m}{\sigma}\sqrt{t}} + 2\sigma^2 t\Phi(\frac{m}{\sigma}\sqrt{t}) \\
+ \quad & m^2t^2\Phi(\frac{m}{\sigma}\sqrt{t}) + O(m\sigma t^{3/2}) \\
= \quad & \sigma^2 t + O(m\sigma t^{3/2}) \qquad (31)
\end{aligned}
$$

The index of dispersions for intervals then is

$$
c_n^2 = \frac{Var(\tilde{U}_{nT})}{nT^2} = \frac{\sigma^2}{T} + O(m\sigma n^{1/2}T^{-1/2}), \quad m \to 0 \qquad (32)
$$

i.e. in the limit $\rho \rightarrow 1$ the IDI becomes independent on $n$ just as a renewal process is.

The only unknown quantity left to find is the diffusion variance $\sigma^2$ as $\rho \rightarrow 1$. To proceed a second order Taylor expansion of the equation

$$z^{(1-1/T)} = B(z) \tag{33}$$

around 1 is needed from which an expansion in $1-\rho$ of the corresponding root $z_\infty$ is possible. From Appendix B it follows that

$$z_\infty = 1 + \frac{2}{\sigma_B^2}(1-\rho) + O((1-\rho)^2) \tag{34}$$

where $\sigma_B^2$ is the variance of the batch size distribution. Since $\sigma^2(\rho) = \frac{2(1-\rho)}{ln(z_\infty)}$ we get

$$\sigma^2(\rho) = \sigma_B^2 + O(1-\rho) \tag{35}$$

From this we see that in the limit $\rho \rightarrow 1$ ($m \rightarrow 0$) the IDI computed from the diffusion formula equals the exact IDI over the entire range of $T$.

# 4 Numerical examples

In this section a series of numerical results are presented to illustrate the behavior of both the exact model as well as the diffusion approximation.

In order to keep the section short we have fixed the interarrival time of the CBR traffic to $T = 20$, as well as the load on the multiplexer, which is fixed to $\rho = 0.8$.

The influence of the burstiness of the background traffic is investigated by using the following 3 batch size distributions:

- Smooth background traffic: *Binomial* distribution with parameters $n$ and $p$ where

$$b(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{36}$$

  In the examples $n = 2$ and $p = 0.375$. For this case the peakedness (the ratio of the variance and mean of the number of arriving cells) is $Z = 0.625$. A binomial batch size distribution is appropriate when the background traffic consists of only a few sources or it may be used to model a background traffic with negative correlations.

- Bursty background traffic: *Generalized negative binomial* (Pascal) distribution with parameters $n$ and $p$ where

$$b(k) = \binom{n+k-1}{k} p^k (1-p)^n \tag{37}$$

  In the examples $n = 0.75$ and $p = 0.5$ with peakedness $Z = 2$. Since a batch of arrivals in a slot in practice always consist of arrivals from different sources a negative binomial distribution is, seen in isolation, not an appropriate batch size distribution. However, applying it anyway provides a simple way to model positive correlations in the background traffic with a model that does not include correlations. This technique cannot be expected to work when the burstiness of the background traffic is very large. However in this case the CBR traffic would possible need some kind of delay priority and a completely different model would be needed.

- *Poisson* background traffic is also used, since it is the classical case. Furthermore, it is an appropriate model for the case where the background traffic consists of a large number of sources.

Figure 3 and 4 depict the effect of the burstiness of the background traffic on the shifted interdeparture time distribution. In Figure 3 it is the distribution between cell 0 and 1 ($n = 1$) while in Figure 4 it is the distribution between cell 0 and 5 ($n = 5$).

In Figure 5 the effect of the load on the shifted interdeparture time distribution can be seen. The background traffic is Poisson ($Z = 1$) and $n = 1$.

By comparing Figure 3 and 5 it can be observed that the burstiness of the background traffic has a stronger influence on the shifted interdeparture time distribution than the load. Since it is the load which can be controlled or estimated, while the burstiness is highly unknown, this gives rise to practical difficulties in the dimensioning of the UPC parameters.

The diffusion approximation performs reasonable well, but for the smooth background traffic case the inaccuracy is slightly larger than for the other cases. The somewhat disappointing result for the Binomial case when $n = 5$ is due to the fact that the steady state waiting time distribution of the D+GI[Binomial]/D/1 deviates more from an exponential distribution that the others.

Figure 6 and 7 depict the distributions of the number of cell departures in a window for the three different burstiness of the background traffic and the load, respectively. Also here we observe the strong effect a variation of the burstiness of the background traffic has on the distribution. The performance of the diffusion approximation is very good, in the Figures the distribution curves almost coincide.

Finally, Figure 8 depicts the index of dispersions for intervals for different burstiness of the background traffic. While some slight inaccuracies are present in the Binomial and Pascal case, it is clearly demonstrated that the actual shape of the IDI curve is very well approximated over the entire range of $n$.

# 5 Conclusion

In this paper we have investigated the cell delay variation in an ATM multiplexer that is fed by the superposition of a CBR stream and a background stream. The cells from the background stream arrive in the form of independent and identically distributed batches with a general batch size distribution.

We have extended the approach of [11] to cope with types of background traffic other than Poisson, and our numerical investigation shows that the variability of the background traffic is a key parameter in the determination of the CDV incurred on the CBR stream at the multiplexer. In our model a moderate correlation of the background traffic can also be taken into account by appropriately adjusting the peakedness of the batch size distribution. For highly correlated background traffic we may apply the method presented in [18].

We have numerically evaluated the diffusion model developed in [3] for other types of background traffic than Poisson. From the numerical results presented here and from our other investigations we conclude that the diffusion approach works reasonable well in general, that the accuracy is highest when the CBR stream is small compared to the background traffic, and when the load is not unreasonable small. Also the peakedness of the background traffic should be kept not too far from 1. Finally, it has also been shown that it is second moment exact in the important heavy traffic limit case for all values of $T$.

The largest advantage of the diffusion model is that it provides a very efficient way to compute the point process characteristics of a CBR stream which have been perturbated by an interfering background stream in an ATM multiplexer.


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## Appendix A: The derivation of the transition matrix

In this Appendix we give the derivation of the transition matrix computation (Eq. 11 - 13).

The elements of the transition matrix can be obtained by

$$q_{j,k} = Q(j,k-1) - Q(j,k) \tag{38}$$

where

$$Q(j,k) = P\{W_i > k \mid W_{i-1} = j\} = \sum_{n=0}^{\infty} P\{W_i > k \mid W_{i-1} = j, N(iT-T, iT) = n\} P\{N(iT-T, iT) = n \mid W_{i-1} = j\} \tag{39}$$

where $N(t_1, t_2) = n$ denotes the event of $n$ arrivals in $]t_1, t_2]$.

The second term of the sum in (39) can be obtained by the $T$-fold convolution of the batch size distribution, i. e.

$$P\{N(iT-T, iT) = n \mid W_{i-1} = j\} = P\{N(iT-T, iT) = n\} = b^{T*}(n) \tag{40}$$

The first factor of each term in the sum of (39) can be derived as follows:

If $j+1 \geq T$ and $n \leq T+k-j-1$ or $j+1 < T$ and $k \geq n$ then

$$P\{W_i > k \mid W_{i-1} = j, N(iT-T, iT) = n\} = 0 \tag{41}$$

If $n > T+k-j-1$ then

$$P\{W_i > k \mid W_{i-1} = j, N(iT-T, iT) = n\} = 1 \tag{42}$$

These formulas are seen by investigating the queue length behaviour, and it is easy to see that the queue length cannot be or must be exceed $k$, respectively.

For the remaining case when $j+1 < T$ and $k < n \leq T+k-j-1$ the Beneš formula, see section 5.3.2. in [8], can be applied:

$$\begin{aligned} &P\{W_i > k \mid W_{i-1} = j, N(iT-T, iT) = n\} = \\ &\quad = \sum_{s=1}^{n-k} P\{N(iT-s, iT) = k+s \mid W_{i-1} = j, N(iT-T, iT) = n\} \times \\ &\quad \times P\{W(iT-s) = 0 \mid W_{i-1} = j, N(iT-s, iT) = k+s, N(iT-T, iT) = n\} \end{aligned} \tag{43}$$

where $W(t)$ denotes the queue length at time $t$.

The first probability in (43) can be derived as follows:

$$\begin{aligned} &P\{N(iT-s, iT) = k+s \mid W_{i-1} = j, N(iT-T, iT) = n\} = \\ &\quad = \frac{P\{N(iT-s, iT) = k+s, N(iT-T, iT) = n\}}{P\{N(iT-T, iT) = n\}} = \\ &\quad = \frac{P\{N(iT-s, iT) = k+s, N(iT-T, iT-s) = n-k-s\}}{P\{N(iT-T, iT) = n\}} = \end{aligned}$$

In the numerator the number of arrivals in the two adjacent windows are independent of each other, therefore

$$\begin{aligned} &= \frac{P\{N(iT-s, iT) = k+s\} P\{N(iT-T, iT-s) = n-k-s\}}{P\{N(iT-T, iT) = n\}} = \\ &= \frac{b^{s*}(k+s) b^{(T-s)*}(n-k-s)}{b^{T*}(n)} \end{aligned} \tag{44}$$

The second probability in (43) can be obtained by using the equivalence with the Beneš analysis of the nD/D/1 queue, see Chapter 6.2.1 in [8] for details, so we have

$$P\{W(iT-s) = 0 \mid W_{i-1} = j, N(iT-s, iT) = k+s, N(iT-T, iT) = n\} = \frac{T-n+k}{T-s} \tag{45}$$

Putting (44) and (45) into (43) we get the first term of (39):

$$\begin{cases} 0 & j+1 \geq T \text{ and } n \leq T+k-j-1 \text{ or} \\ & \quad j+1 < T \text{ and } k \geq n \\ 1 & n > T+k-j-1 \\ \displaystyle\sum_{s=1}^{n-k} \frac{T-n+k}{T-s} \times & \\ \quad \times \frac{b^{s*}(k+s) b^{(T-s)*}(n-k-s)}{b^{T*}(n)} & j+1 < T \text{ and } k < n \leq T+k-j-1 \end{cases} \tag{46}$$

Finally, introducing $P_n(j,k)$ as the product of the two terms in (39) we have

$$Q(j,k) = \sum_{n \geq 0} P_n(j,k) \tag{47}$$

with

$$P_n(j,k) = \begin{cases} 0 & j+1 \geq T \text{ and } n \leq T+k-j-1 \text{ or} \\ & \quad j+1 < T \text{ and } k \geq n \\ b^{T*}(n) & n > T+k-j-1 \\ \displaystyle\sum_{s=1}^{n-k} b^{s*}(k+s) \times & \\ \quad \times b^{(T-s)*}(n-k-s) \frac{T-n+k}{T-s} & j+1 < T \text{ and } k < n \leq T+k-j-1 \end{cases} \tag{48}$$

# Appendix B: The Taylor expansion of the dominant root

For completeness we make a third order analysis which enables a first order formula for the diffusion variance.

A third order Taylor expansion of $z^{1-1/T} = B^*(z)$ around $z = 1$ yields

$$1 + A_1(z-1) + A_2(z-1)^2 + A_3(z-1)^3 = 1 + B_1(z-1) + B_2(z-1)^2 + B_3(z-1)^3 \tag{49}$$

where

$$A_1 = (1 - \frac{1}{T}), \quad A_2 = \frac{-\frac{1}{T}(1-\frac{1}{T})}{2}, \quad A_3 = \frac{\frac{1}{T}(1-(\frac{1}{T})^2)}{6} \tag{50}$$

and

$$B_1 = E(B), \quad B_2 = \frac{E(B^2) - E(B)}{2} \quad B_3 = \frac{E(B(B-1)(B-2))}{6} \tag{51}$$

Now

$$B_1 = E(B) = \rho - \frac{1}{T} = 1 - \frac{1}{T} - (1-\rho) =: b_1^0 + b_1^1(1-\rho) \tag{52}$$

where $b_1^0 = 1 - 1/T$ and $b_1^1 = -1$. Similarly

$$\begin{aligned} B_2 &= \frac{E(B^2) - E(B)}{2} = \frac{(1-\frac{1}{T})(c_B - \frac{1}{T})}{2} - \frac{(1 + c_B - \frac{2}{T})}{2}(1-\rho) + \frac{1}{2}(1-\rho)^2 \\ &=: b_2^0 + b_2^1(1-\rho) + O((1-\rho)^2) \end{aligned} \tag{53}$$

where $c_B = \frac{Var(B)}{E(B)}$ is the peakedness of the Batch size distribution. Finally,

$$\begin{aligned} B_3 &= \frac{E(B(B-1)(B-2))}{6} = \frac{\mu_{3(\rho=1)} - (1-\frac{1}{T})(3c_B + 1 - \frac{1}{T})}{6} + \frac{O(1-\rho)}{6} \\ &=: b_3^0 + O(1-\rho) \end{aligned} \tag{54}$$

where $\mu_3$ is the third moment of the batch size distribution. A third order Taylor expansion of the root $z$ is

$$z_\infty(1-\rho) = 1 + r_1(1-\rho) + r_2(1-\rho)^2 + r_3(1-\rho)^3 \tag{55}$$

Insertion of (52), (53), (54) and (55) in (49) yields:

$$\begin{aligned} A_1 r_1(1-\rho) &+ (A_1 r_2 + A_2 r_1^2)(1-\rho)^2 + (A_1 r_3 + 2A_2 r_1 r_2 + A_3 r_1^3)(1-\rho)^3 \\ &= b_1^0 r_1(1-\rho) + (b_1^0 r_2 + b_1^1 r_1 + b_2^0 r_1^2)(1-\rho)^2 \\ &+ (b_1^0 r_3 + b_1^1 r_2 + 2b_2^0 r_1 r_2 + b_2^1 r_1^2 + b_3^0 r_1^3)(1-\rho)^3 \end{aligned} \tag{56}$$

Equalizing coefficients corresponding to $(1-\rho)$ gives

$$A_1 = b_1^0 \tag{57}$$

which is fulfilled since $A_1 = 1 - \frac{1}{T} = b_1^0$. From equalizing coefficients corresponding to $(1-\rho)^2$ we get

$$r_1 = \frac{b_1^1}{A_2 - b_2^0} \tag{58}$$

Finally, equalizing third order coefficients gives

$$r_2 = \frac{b_2^1 + (b_3^0 - A_3) r_1}{2(A_2 - b_2^0) r_1 - b_1^1} r_1^2 \tag{59}$$

which after insertion of (58) gives

$$r_2 = \frac{b_1^1(A_3 b_1^1 - A_2 b_2^1 + b_2^0 b_2^1 - b_1^1 b_3^0)}{(b_2^0 - A_2)^3} \tag{60}$$

From (50), (52) and (53) we end up with

$$z_\infty(1-\rho) = 1 + \frac{2}{(1-\frac{1}{T})c_B}(1-\rho) + r_B(\mu_3, c_B)(1-\rho)^2 + O((1-\rho)^3) \tag{61}$$

where

$$r_B(\mu_3, c_B) = \frac{\frac{2}{3}(3c_B + 1 + \frac{1}{T^2} - \frac{T}{T-1}\mu_3) - 2(1 + c_B - \frac{2}{T})c_B}{(1-\frac{1}{T})c_B^3} \tag{62}$$

Since

$$\begin{aligned} \sigma^2 &= \frac{2(1-\rho)}{ln(z_\infty(1-\rho))} = \frac{2(1-\rho)}{r_1(1-\rho) + (r_2 - \frac{1}{2}r_1^2)(1-\rho)^2 + O((1-\rho)^3)} \\ &= \frac{2}{r_1} - (\frac{2r_2}{r_1^2} - 1/2)(1-\rho) + O((1-\rho)^2) \end{aligned} \tag{63}$$

we end up with

$$
\begin{aligned}
\sigma^2 &= (1-\frac{1}{T})c_B \\
&- (\frac{\frac{2}{3}(3c_B+1+\frac{1}{T^2}-\frac{T}{T-1}\mu_3)-(2+c_B-\frac{2}{T})c_B}{c_B})(1-\rho)+O((1-\rho)^2) \\
&= \sigma_B^2+O(1-\rho) \qquad (64)
\end{aligned}
$$

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