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THE NATURE OF DISCRETE SECOND-ORDER SELF-SIMILARITY

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Abstract

A new treatment of second-order self-similarity and asymptotic self-similarity for stationary discrete time series is given, based on the fixed points of a renormalisation operator with normalisation factors which are not assumed to be power-laws. A complete classification of fixed points is provided, consisting of the fractional noise and one other class. A convenient variance-time function approach to process characterisation is used to exhibit large explicit families of processes asymptotic to particular fixed points. A natural, general definition of discrete long-range dependence is provided and contrasted with common alternatives. The closely related discrete form of regular variation is defined, its main properties given, and its connection to discrete self-similarity explained. Folkloric results on long-range dependence are proved and/or disproved rigorously.

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1. Introduction

In this paper we reexamine the notion of second-order self-similarity and asymptotic self-similarity in discrete time series. These concepts are frequently used in applications, and it is commonly supposed that they are as well understood as in the continuous time case. In fact, existing treatments operate from a perspective of considering discrete self-similarity as a simple 'sampling' of the continuous. There is a need to define operations, pose questions, and prove results rigorously for the discrete case in its own right, to explore the possible existence of new, intrinsically discrete effects, and to eliminate procedures and misconceptions related to thinking in continuous terms. More specifically, we revisit the subject with three main objectives.

1. To provide a natural and inclusive definition of second-order self-similarity (*SS*) specific to discrete time series. The definition proposed in section 2.1 is novel in that it is not

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assumed that normalisation factors must be power-laws. In section 3 we give a complete classification of the set of *SS* processes under this definition, which is larger than the familiar *fractional noise* processes because of intrinsically discrete effects.

2. To provide a natural definition of second-order asymptotic self-similarity (*ASS*), and an associated functional definition (based on the variance-time function) which is very useful for calculations. Using it, we exhibit large explicit families of *ASS* processes associated to particular self-similar processes, with an emphasis on those which have received little attention in the past. Section 4 deals with this material.

3. To clarify issues and prove results regarding long-range dependence (LRD), an important class of second-order asymptotic self-similar processes. In section 5 we provide a new definition based on the discrete framework, and compare it to related definitions in the literature. We provide proofs, or disprove, certain 'folklore' theorems, and clarify the closely related misconceptions regarding discrete forms of *regular variation* (2). In doing so, we bring together and prove a number of results regarding second-order self-similarity which are scattered in the literature, and whose proofs are difficult or impossible to find.

Although the mathematical literature on self-similarity is well developed, it is dominated by continuous processes. The extensive 1987 review of Vervaat (16), is based on the continuous *SS* concept, as is the fundamental work of Lamperti (9) in 1962. Many generalisations of self-similarity, and alternative definitions of long-range dependence, have already been considered. These include self-similarity in higher dimensions (15; 10), and non-Gaussian classes both with and without second order moments (15; 10; 14; 8). Power-law normalisation factors however are essentially universally used, even for studies of discrete processes, a practice clearly inspired by the known power-law nature of continuous time self-similarity. Our definition is based on the fixed point formulation introduced by Sinai (15) for discrete processes in 1976. This approach is in many respects very general and is widely used. For example it is equivalent to the definition employed in the review of (4), which is frequently cited in applications in time series. We generalise this classic definition by removing the unnecessary restriction to power-law normalisation factors. This relaxation bears fruit in the discrete context and brings greater insight.

2. Renormalisation and Discrete Self-Similarity

Let $\{X(t),\, t \in \mathbb{Z}\}$ denote a discrete time second-order stationary stochastic process. The mean μ and variance $\mathcal{V}$ of such a process are independent of t, and the *autocovariance function*, $\gamma(k) := E[(X(t)-\mu)(X(t+k)-\mu)]$, depends only on the lag k, $k \in \mathbb{Z}$, and $\gamma(k) = \gamma(-k)$. We will assume that the process is non-trivial, that is $\mathcal{V} > 0$. By dividing out this size parameter we obtain the *autocorrelation function* (ACF): $\rho(k) := \frac{\gamma(k)}{\gamma(0)} = \frac{\gamma(k)}{\mathcal{V}}$, which defines the nature of the process. Apart from requiring a finite positive variance, no assumption is made on the form of $\gamma(k)$, nor on the distributions of the process. Technically this paper deals with functional relationships satisfied by general covariance functions, and random variables play no essential role.

In addition to the familiar descriptors of second order structure above, it turns out to be fruitful to work with an equivalent pair of functions, the *variance time function* and its normalised form, the *correlation time function* (CTF). The variance time function is defined as $\omega(n) = \sum_{k=0}^{n-1} \sum_{i=-k}^{k} \gamma(i) = n\gamma(0) + 2\sum_{i=1}^{n-1} i\gamma(n-i)$, $n = 1, 2, 3\cdots$, while the CTF is just $\phi(n) = \frac{\omega(n)}{\omega(1)} = \frac{\omega(n)}{\mathcal{V}}$. In terms of the original process, $\omega(n)$ is just the variance of the sum $\sum_{t=1}^{n} X(t) = n\bar{X}$, where $\bar{X}$ is the sample mean. Just as ω can be expressed uniquely in terms of γ as $\omega = \mathbf{I}\{\gamma\}$, where $\mathbf{I}$ is the operator defined by the double sum above, γ can be written as $\gamma = \mathbf{D}\{\omega\}$ via a double-differencing operator $\mathbf{D}$, defined as:

$$\mathbf{D}_i\{f(i)\}(n) = \begin{cases} f(1) & : n = 0 \\ \frac{1}{2}(f(2) - 2f(1)) & : n = 1 \\ \frac{1}{2}(f(n+1) - 2f(n) + f(n-1)) & : n > 1. \end{cases} \tag{1}$$

It is easy to confirm that these operators are each one-to-one and are inverses of each other, so the autocovariance function and the variance time function are entirely equivalent, and it is obvious that the normalised versions, the ACF and the CTF, are likewise entirely equivalent. Another equivalent alternative is the covariance sum function, $S(n) = \sum_{k=-n}^{n} \gamma(k)$, $n \geq 0$, which will play a role in sections 4 and 5. Note that $w(1) = S(0) = \gamma(0) = \mathcal{V}$, so $\phi(1) = \rho(0) = 1$.

Self-similarity relates to invariance with respect to a rescaling operation. In the present context, this decomposes into a rescaling over time, followed by a rescaling of amplitude. For the first of these the standard approach, which we adopt here, is that of *aggregation*. For a fixed $m \geq 1$ the *aggregation of level* m of the original process X is the process $X^{(m)}$ defined as

$$X^{(m)}(t) := \frac{1}{m} \sum_{j=m(t-1)+1}^{mt} X(j). \tag{2}$$

The normalisation factor $1/m$ is conventionally present in definitions of aggregation. As it will be overwritten by the subsequent amplitude rescaling described below, its role is purely incidental and it may be ignored. The $\gamma, \rho, \omega, \phi$ functions and the variance of the m-aggregated process will be denoted by $\gamma^{(m)}$, $\rho^{(m)}$, $\omega^{(m)}$, $\phi^{(m)}$ and $\mathcal{V}^{(m)}$ respectively.

Understanding the effect of aggregation on the covariance structure amounts to expressing $\gamma^{(m)}$ in terms of γ. Beginning with the definition: $\gamma^{(m)}(k) = E[X^{(m)}(0)X^{(m)}(k)]$, and expanding $X^{(m)}(0)$ and $X^{(m)}(k)$ according to equation (2), we obtain

$$\gamma^{(m)}(k) = \frac{1}{m^2}\left[m\gamma(km) + \sum_{i=1}^{m-1} i(\gamma((k-1)m+i) + \gamma((k+1)m-i))\right]. \tag{3}$$

To express $\omega^{(m)}$ in terms of ω we see from equation (3) that

$$\mathcal{V}^{(m)} = \gamma^{(m)}(0) = \frac{1}{m^2}\left(m\gamma(0) + 2\sum_{i=1}^{m-1} i\gamma(m-i)\right) = \frac{\omega(m)}{m^2}, \tag{4}$$

and so $\mathcal{V}^{(mn)} = \frac{\omega(mn)}{(mn)^2}$. But $\mathcal{V}^{(mn)}$ is the variance of the mn-aggregated process, which is also the n-aggregation of the m-aggregation, and so using equation (4) again we obtain $\mathcal{V}^{(mn)} = \mathcal{V}^{(m)(n)} = \frac{\omega^{(m)}(n)}{n^2}$, from which it follows that

$$\omega^{(m)}(n) = \frac{\omega(mn)}{m^2}. \tag{5}$$

After a m-aggregation rescaling in time, we need to rescale in amplitude. This is performed naturally by dividing by the new variance $\mathcal{V}^{(m)}$, which amounts to examining the effect of aggregation on the correlation structure. Combining the two rescalings by normalising equations (3) and (5) leads to two functional incarnations of the renormalisation operator we consider.

Definition 1. (*Renormalisation Operator.*) Given a process X with ACF ρ and CTF ϕ, the m-renormalisation consists of a m-aggregation followed by normalisation with $\mathcal{V}^{(m)}$. The renormalised functions are given respectively by

$$\rho^{(m)}(k) = \left[m\rho(km) + \sum_{i=1}^{m-1} i(\rho((k-1)m+i) + \rho((k+1)m-i))\right] / \left[m\rho(0) + 2\sum_{i=1}^{m-1} i\rho(n-i)\right] \tag{6}$$

and

$$\phi^{(m)}(n) = \frac{\phi(mn)}{\phi(m)}. \tag{7}$$

The greater simplicity of the correlation time formulation is apparent.

There exist γ with $\mathcal{V} > 0$ for which $\mathcal{V}^{(m)} = 0$ for some m, so that the corresponding $\rho^{(m)}$ and $\phi^{(m)}$ are not defined. For example the non-ergodic process $\{\cdots, Z, -Z, Z, -Z, Z\cdots\}$, where Z is a symmetric random variable with unit variance, has periodic $\gamma = \{1, -1, 1, -1, 1, \ldots\}$ (beginning at lag $k = 0$), $S = \{1, -1, 1, -1, 1, \ldots\}$ and $\omega = m^2\mathcal{V}^{(m)} = \{1, 0, 1, 0, 1, \ldots\}$. Consider a sequence of aggregations $m = 1, 2, 3\cdots$ of an initial process. There are two such problematic cases: (i) there exist an $m' \geq 0$ such that $\mathcal{V}^{(m)} = 0$ for all $m \geq m'$, (ii) zero and non-zero values of $\mathcal{V}^{(m)}$ occur infinitely often as $m \to \infty$. As in neither of these cases does the sequence have a meaningful, non-degenerate limit, we exclude them from the fixed point development below without loss of generality.

2.1. Self-Similarity

Although different definitions of second-order self-similarity can be found in the literature, they share the common idea of processes which do not change their qualitative statistical behaviour after aggregation. The most natural way to capture this is to directly define a self-similar process as one whose second order structure is a fixed point of the renormalisation operator.

Definition 2. (*Second-order Self-Similarity.*) A process is second-order self-similar if $\rho^{(m)} \equiv \rho$, or equivalently $\phi^{(m)} \equiv \phi$, for all $m = 1, 2, 3, \cdots$.

This definition is not novel as such, for example it is used in (15; 4). What is new however is allowing the amplitude rescaling factors $\mathcal{V}^{(m)}$ to take their natural values, which are dependent on the process under study, rather than restricting them according to desired or preconceived notions of their asymptotic form. The equivalence of the ρ and ϕ formulations is an immediate consequence of the fact that the operators $\mathbf{I}$ and $\mathbf{D}$ are inverses.

The only examples in the literature of processes obeying this definition are those with the same second order behaviour as fractional Gaussian noise. We call these the *fractional noise* (FN_H) class, defined by $\rho_{\text{FN}}(k) = \mathbf{D}_i\{i^{2H}\}(k)$, or equivalently by the simpler $\phi_{\text{FN}}(n) = n^{2H}$, where $H \in [0, 1]$ is the Hurst parameter (14). The extreme cases $H \in \{0, 1\}$ are sometimes excluded. We include them for completeness, and pay particular attention to the interesting $H = 0$ case in section 4. To study processes satisfying the definition more generally, we exploit the simplicity of the CTF formulation and set $\phi^{(m)} = \phi$ in equation (7) to obtain the fixed point equation

$$\phi(nm) = \phi(n)\phi(m), \tag{8}$$

for which it is trivial to show that $\phi_{\text{FN}}(n) = n^{2H}$ is a solution. In (15) it is proved that if the power-law normalisation factors are selected instead of $\mathcal{V}^{(m)}$, then FN_H is the only possible class of fixed point processes. Since $\mathcal{V}^{(m)} = \mathcal{V}m^{2H-2}$ for FN_H , the two definitions (power-law or variance based normalisation) are equivalent in this important but special case. Note that from equation (8), the set of fixed points is identical to that of totally multiplicative functions ((12), p.308).

It is easily verified that the general solution of equation (8) can be written as

$$\phi(m) = \prod_{i=1}^{s} \phi^{r_i}(p_i), \quad \text{for each } m = \mathbb{Z}^+, \tag{9}$$

where the p_i are the s distinct prime factors of m, and r_i is the multiplicity of p_i. It is therefore clear that a huge variety of solutions exist which are different, even radically so, from simple power-laws. For example one could even assign the values of ϕ at primes randomly, since at primes there are no constraints imposed by (9) to satisfy, so the function may take any value. In contrast, at non-primes the function is determined by the choices made on its prime factors. It is apparent however that of these fixed point solutions, a great many cannot correspond to valid processes. For example $\phi(n) = n^{\alpha}$ is a solution for all real α, which for $\alpha \in [0, 2]$ is just fractional noise. However $\alpha > 2$ implies that the variance of the sum of m copies of $X(t)$ is greater than $m^2\mathcal{V}$, which is impossible. The missing ingredient is the need for the fixed point solution to also be positive semi-definite.

Recall that a function $f(k)$ defined on $k = 0, 1, 2, \ldots$ is said to be *positive semi-definite* if for any $n = 1, 2, \ldots$ and for any real vector $\underline{\mathbf{a}}$ of length n

$$\sum_{1 \leq i,j \leq n} a_i f(|i-j|) a_j \geq 0.$$

It can be shown that a necessary and sufficient condition for a function to be the autocovariance function of a process is that it be positive semi-definite (3). In order to have a uniform notation across the two formulations, we introduce the following term.

Definition 3. (*Valid functions.*)
A function γ is said to be a *valid* (autocovariance or autocorrelation) function if it is positive semi-definite.
A function ω is said to be a *valid* (variance time or correlation time) function if $\gamma = \mathbf{D}\{\omega\}$ is positive semi-definite.

The requirement that ϕ be valid imposes strong constraints on the solutions of equation (8). It can however be very difficult to judge if a given function is positive semi-definite, and therefore the existence of new fixed point solutions which are simultaneously valid processes is far from obvious. Figure 1 shows two examples of ρ whose corresponding ϕ functions satisfy equation (8). Despite obeying the necessary condition $\rho(k) \in [-1, 1]$ for all k, it will be shown that the function on the left is not valid.

3. Classification of Self-Similar Processes

In this section we provide a complete characterisation of discrete second-order self-similar processes. We first recall the following two parameter class of functions, introduced in (7), which are fixed point solutions of equation (8) through the general prime number construction of equation (9).

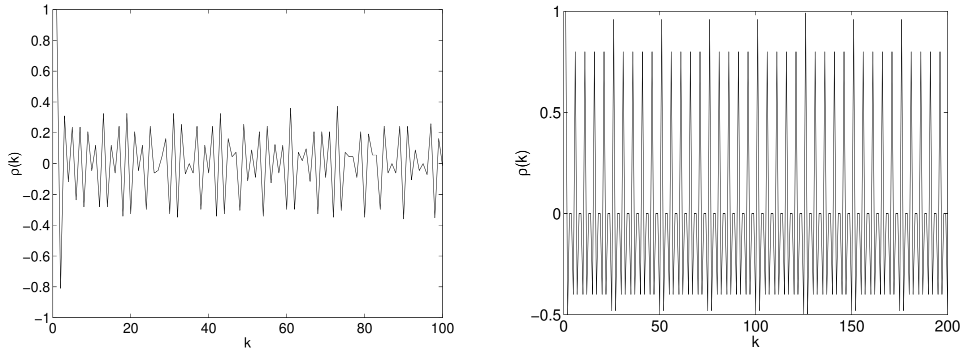

FIGURE 1: Examples of ρ functions corresponding to simple non-FN$_H$ fixed points with $\rho(k) \in (-1, 1)$, $k \neq 0$. Left: 100 lags of an invalid fixed point defined by $\phi(p) = 0.38$ for each prime p. Right: 200 lags of a valid fixed point from the (Almost Periodic) AP$_{q,c}$ family with $q = 5$, $c = 0.2$.

Definition 4. (*The almost periodic fixed point family AP$_{q,c}$.*) The two parameter family of fixed points defined by $\phi(1) = 1$, $\phi(p) = 1$ for all primes p except $p = q$, where $\phi(q) = c$, $c \in (0, 1)$, are called *Almost Periodic*, and denoted by AP$_{q,c}$.

A member of the family is exhibited on the right in figure 1, where its 'almost periodic' nature is readily appreciated. That it is not periodic can be seen from the fact that a new value of the function is reached for the first time whenever n reaches the next higher power of q. Note that both ρ and ϕ oscillate and therefore fail to have limits at infinity. The following theorem was recently proved in (7).

Theorem 1. (New SS processes.) *Each member of the AP$_{q,c}$ family is a valid fixed point.*

The next theorem, our main result, completes the classification of valid fixed points by showing for the first time that the two valid classes thus identified, FN$_H$, and AP$_{q,c}$, are in fact the only possibilities. Two consequences of positive semi-definiteness will be used (proofs in the appendix) to exclude classes of fixed points from consideration.

CTF–(i) $\phi(m - n) \leq 2(\phi(m) + \phi(n))$ for all $m, n \in \mathbb{Z}^+$, $m - n \geq 1$.

CTF–(ii) For any $n \in \mathbb{Z}^+$, $|S(n)/\mathcal{V}| < 2\sqrt{\phi(n)} + C$, where C is a constant independent of n.

The following concept is at work throughout the proof. For any fixed point ϕ, the powers of each prime p lie along a unique power-law curve given by $f(x) = x^{\alpha_p}$, where $\alpha_p = \frac{\ln \phi(p)}{\ln p}$. Hence if $p > q$ and $\alpha_p > \alpha_q$, then $\phi(p) > \phi(q)$.

Theorem 2. (SS Process Classification.) *The set of valid fixed points is given by the union of the FN$_H$ and AP$_{q,c}$ families.*

Proof.
Case I: $\phi(p) > 1$ for some prime p. This implies that $\alpha := \sup_p \alpha_p > 0$. If the α_p are all equal the fixed point is of power-law type, and we already know that $\alpha \in (0, 2]$ is the range of validity, which is just FN$_H$ with $H \in (0, 1]$. Assume then that the α_p are not all equal.

Case Ia: The supremum α is not attained. Hence there exists an infinite sequence of primes p_i s.t.

$$\alpha_{p_i} > \max_{q<p_i} \alpha_q.$$

Let $\beta_i := \alpha_{p_i}$. According to **CTF**–(ii), and using the fact that $\phi(m) < \phi(p_i)$ for all $m < p_i$,

$$\phi(p_i^2) - \phi(p_i^2 - 1) = S(p_i^2 - 1)/\mathcal{V} < 2\sqrt{\phi(p_i)^2} + C = 2\phi(p_i) + C = 2p_i^{\beta_i} + C. \quad (10)$$

For $p_i > 2$ clearly $2|(p_i^2 - 1)$, and using the definition of β_i we have

$$\phi(p_i^2 - 1) = \phi(\frac{p_i^2 - 1}{2})\phi(2) < \left(\frac{p_i^2}{2}\right)^{\beta_i} \phi(2) = p_i^{2\beta_i} \frac{\phi(2)}{2^{\beta_i}}. \quad (11)$$

Combining equations (10) and (11) we have

$$p_i^{2\beta_i} \left(1 - \frac{\phi(2)}{2^{\beta_i}}\right) < \phi(p_i^2) - \phi(p_i^2 - 1) < 2p_i^{\beta_i} + C.$$

Dropping the middle term, substituting $\gamma = 1 - \frac{\phi(2)}{2^{\beta_i}} > 0$, and dividing by $p_i^{\beta_i}$ we obtain

$$p_i^{\beta_i} < \frac{1}{\gamma}\left(2 + \frac{C}{p_i^{\beta^i}}\right). \quad (12)$$

Since $\beta_i > 0$, as $i \to \infty$ the l.h.s. of (12) diverges, whereas the r.h.s. is bounded, a contradiction. Such cases are therefore invalid.

Case Ib: The supremum α is attained. Let q be a prime with $\alpha_q = \alpha$, and p another with $\alpha_p < \alpha$. According to the Euler theorem (12), there exists a positive integer j such that $q^j = 1 \mod p$, and so for any $k \in Z^+$ $q^{kj} = 1 \mod p$. Using this, and by the definition of α,

$$\phi(q^{kj} - 1) = \phi(p)\phi\left(\frac{q^{kj} - 1}{p}\right) \le \phi(p)\left(\frac{q^{kj} - 1}{p}\right)^\alpha < q^{kj\alpha}\left(\frac{\phi(p)}{p^\alpha}\right).$$

Combining this with property **CTF**–(ii) we obtain

$$q^{kj\alpha}\left(1 - \frac{\phi(p)}{p^\alpha}\right) < \phi(q^{kj}) - \phi(q^{kj} - 1) < 2q^{kj\alpha/2} + C,$$

which, similarly to equation (12), yields a contradiction as $k \to \infty$. Such cases are therefore invalid.

Case II: $\phi(p) \le 1$ for all p prime. The case of $\phi = 1$ is just FN_0 and is valid. The case of only one prime q such that $\phi(q) < 1$ is included in $\mathrm{AP}_{q,c}$ and is valid. Assume then that there exists two different primes q_1 and q_2 for which $\phi(q_i) < 1$. This includes power-law fixed points with $\alpha < 0$ ($H < 0$).

Using the non-negativity of ϕ and substituting $k = m - n$ in **CTF**–(i), we have

$$0 \le \phi(k) \le 2(\phi(n + k) + \phi(n)).$$

For any $\epsilon > 0$ we can chose an $r \in \mathbb{Z}^+$ sufficiently large such that $\phi(q_i^r) < \epsilon$, for $i = 1, 2$. It is easy to see that for any $k \ge 1$ it is possible to find an n such that $q_1^r|n$ and $q_2^r|(n + k)$. Since $\phi(p) \le 1$ for all p, this implies that for each k

$$0 \le \phi(k) \le 2(\phi(n + k) + \phi(n)) \le 4\epsilon$$

which is only consistent with the fixed point $\phi = 0$, which is excluded as $\mathcal{V} > 0$. All such cases are therefore invalid.

Finally we note the following alternative approach to prove the main direction of this result, that the set of self-similar processes is no larger than the union of the FN_H and $\mathrm{AP}_{q,c}$ families. It can be shown that if ϕ does not vanish, then $\sqrt{\phi}$ can be extended to a valuation of the field of rational numbers, and then Ostrowski's classification of such valuations (see for instance Narkiewicz (11)) can be applied.

4. Asymptotically Self-Similar Processes

Consider some initial ρ for which the aggregation sequence $\{\rho^{(m)}\}$ exists and converges pointwise to a *limit point* ρ^*. The set of initial ρ which have ρ^* as their limit will be called the *domain of attraction* of ρ^*. If ρ^* is a self-similar process, then it is natural to identify its domain of attraction as the class of asymptotically self-similar processes corresponding to it. The idea of defining *ASS* via some kind of convergence to *SS* processes, under some renormalisation operation, is nearly universal, and pointwise convergence is very commonly used (for instance in (4)).

To give a definition of asymptotic self-similarity in the same spirit as definition 2 for self-similarity, we must first show that the equivalence of ρ and ϕ, which was obvious for the fixed points themselves, carries over to limit operations. To do so, it is convenient to adopt an operator notation for the manipulation of functions which are not necessarily valid. This allows the relationships between limit points and fixed points to be determined in an elegant way. In section 4.2, we take advantage of the simplicity of the CTF formulation to provide non-trivial classes of asymptotically self-similar processes in several interesting and important cases.

4.1. Limit Points, Fixed Points, and Equivalence of ϕ and ρ

First recall from section 2 the operators $\mathbf{I}$ and $\mathbf{D}$ that perform double Integration and double Differencing respectively. Although previously used to map valid functions according to $\omega = \mathbf{I}\{\gamma\}$ and $\gamma = \mathbf{D}\{\omega\}$, in this section we let $\gamma : \mathbb{Z} \mapsto \mathbb{R}$ and $\omega : \{1, 2, \cdots\} \mapsto \mathbb{R}$ be arbitrary normalisable functions, that is, $\gamma(0) > 0$ and $\omega(1) > 0$.

The normalisation operator $\mathbf{N}$ acts by simply dividing by the 'first' element of its function argument. This amounts to dividing by $\mathcal{V} \neq 0$ when operating on either of γ or ω, (we allow the difference in index, 0 for γ and 1 for ω, in a harmless abuse of notation). The aggregation operators acting on γ and ω, according to equations (3) and (5) respectively, are denoted by $\mathbf{G_m}$ and $\mathbf{W_m}$. Finally the renormalisation operators, acting on ρ and ϕ according to equations (6) and (7) respectively, are denoted by $\mathbf{R_m} = \mathbf{NG_m}$ and $\mathbf{P_m} = \mathbf{NW_m}$. Thus we say that the function ρ^* is a fixed point of the operator family $\{\mathbf{R_m}\}$ if, for all $m = 1, 2, 3, \cdots, \mathbf{R_m}\{\rho^*\} \equiv \rho^*$, and similarly for a fixed point ϕ^*.

Definition 5. (*Limit points of* $\mathbf{R_m}$ *and* $\mathbf{P_m}$.)
The function ρ^* is a limit point under $\{\mathbf{R_m}\}$ of some function γ, if $\mathbf{R_m}\{\gamma\} \to \rho^*$ pointwise as $m \to \infty$.
The function ϕ^* is a limit point under $\{\mathbf{P_m}\}$ of some function ω, if $\mathbf{P_m}\{\omega\} \to \phi^*$ pointwise as $m \to \infty$.

The definition of domains of attraction carries over in the obvious way.

The following operator properties are straightforward to prove.

Lemma 1. (Operator properties.)
(a) $\mathbf{N}\{c\gamma\} = \mathbf{N}\{\gamma\}$, *where* γ *is a normalisable function and* $c \neq 0$ *a real factor.*

(b) $\mathbf{I}$ *and* $\mathbf{D}$ *are inverses of each other.*
(c) $\mathbf{I}$ *and* $\mathbf{D}$ *preserve normalisation, that is* $\phi \equiv \mathbf{I}\{\rho\}$ *and* $\rho \equiv \mathbf{D}\{\phi\}$.
(d) $\mathbf{I}$ *and* $\mathbf{D}$ *commute with* $\mathbf{N}$ *and with multiplication by a constant.*
(e) $\mathbf{IG_m} = \mathbf{W_m I}$.
(f) Let $\{N_j\}$ *be a sequence of real functions that converge pointwise to a function* N*, i.e.* $\lim_{j\to\infty} N_j(k) = N(k)$ *for each* k*. Then, letting* $\mathbf{T}$ *be any of* $\mathbf{I}$, $\mathbf{D}$, $\mathbf{G_m}$ *or* $\mathbf{W_m}$,

$$\mathbf{T}\{\lim_{j\to\infty} N_j\} \equiv \lim_{j\to\infty} \mathbf{T}\{N_j\}.$$

The following theorem established the link between the limit points of ρ and ϕ and thereby their equivalence for the study of asymptotically self-similar processes.

Theorem 3. (Equivalence of limit points.)
The limit point of γ *under* $\mathbf{R_m}$ *is* ρ^* *if and only if the limit point of* $\omega \equiv \mathbf{I}\{\gamma\}$ *under* $\mathbf{P_m}$ *is* $\phi^* \equiv \mathbf{I}\{\rho^*\}$.

Proof. Let $\lim_m$ denote $\lim_{m\mapsto\infty}$, and assume that $\lim_m \mathbf{R_m}\{\gamma\} \equiv \rho^*$. Operating with $\mathbf{I}$ yields $\mathbf{I}\lim_m \mathbf{R_m}\{\gamma\} \equiv \mathbf{I}\{\rho^*\} \equiv \phi^*$. Now $\mathbf{I}\lim_m \mathbf{R_m}\{\gamma\} \equiv \mathbf{I}\lim_m \mathbf{NG_m}\{\gamma\} \equiv \lim_m \mathbf{ING_m}\{\gamma\}$ using lemma 1(f), and from (d) this becomes $\lim_m \mathbf{NIG_m}\{\gamma\} \equiv \lim_m \mathbf{NW_m I}\{\gamma\} \equiv \lim_m \mathbf{NW_m}\{\omega\} \equiv \lim_m \mathbf{P_m}\{\omega\}$. Thus $\lim_m \mathbf{P_m}\{\omega\} = \phi^*$ as required. A similar proof holds for the other direction.

With the equivalence of the two formulations established even for limit operations, we are now free to select the domain of investigation and carry over the results to the other.

Theorem 4. (Limit points must be fixed points.)
(ACF) If ρ^* *is a limit point of some* γ *under* $\mathbf{R_m}$*, then it is also a fixed point of* $\mathbf{R_m}$.
(CTF) If ϕ^* *is a limit point of some* ω *under* $\mathbf{P_m}$*, then it is also a fixed point of* $\mathbf{P_m}$.

Proof. We use the ACF definition. We have to show that for each m and n fixed, $\rho^{*(m)}(n) = \rho^*(n)$. Now $\rho^{*(m)} \equiv \mathbf{R_m}\lim_l \mathbf{R_l}\{\gamma\} \equiv \mathbf{NG_m}\lim_l \mathbf{R_l}\{\gamma\} \equiv \mathbf{N}\lim_l \mathbf{G_m R_l}\{\gamma\}$ from lemma 1(f). This can be written as $\mathbf{N}\lim_l c_{m,l}\mathbf{R_m R_l}\{\gamma\} \equiv \mathbf{N}\lim_l c_{m,l}\mathbf{R_{lm}}\{\gamma\}$ where the γ-dependent constants $c_{m,l}$ converge: $\lim_l c_{m,l} = c_m$. Thus $\mathbf{N}\lim_l c_{m,l}\mathbf{R_{lm}}\{\gamma\} \equiv \mathbf{N}\{\lim_l c_{m,l}\lim_l \mathbf{R_{lm}}\{\gamma\}\} \equiv \mathbf{N}\{c_m\rho^*\} = \rho^*$, using lemma 1(a), and the fact that a subsequence of a convergent sequence is convergent to the same limit.

This result tells us that the set of fixed points is very important. Any initial function γ either tends to a fixed point under renormalised aggregation, or it does not converge at all.

4.2. Classes of Asympotically Self-Similar Processes

We begin with a new definition of *ASS* processes based on ϕ corresponding to definition 2.

Definition 6. (*Asymptotic second-order self-similarity.*) A process is asymptotically second-order self-similar if its correlation time function satisfies

$$\lim_{m\to\infty} \mathbf{P_m}\phi = \lim_{m\to\infty} \phi^{(m)} \to \phi^*, \quad \text{or equivalently} \quad \lim_{m\to\infty} \frac{\phi(nm)}{\phi(m)} = \phi^*(n) \tag{13}$$

for all $n = 1, 2, 3\cdots$, where ϕ^* is a fixed point.

The following lemma confirms our intuition that the fixed point ϕ^* must be valid.

Lemma 2. (Limits of processes remain processes.) *If a sequence $\{\omega_k\}$ of valid variance time functions converges pointwise, then the limit ω is also a valid variance time function.*

Proof. Let $\gamma_k :\equiv \mathbf{D}\omega_k$ and $\gamma :\equiv \mathbf{D}\omega$. Since $\gamma_k(i)$ for any finite k is a finite linear combination of ω values, the pointwise convergence of ω_k implies that of γ_k. For any finite k, and any vector $\underline{\mathbf{a}}$ of finite length m, $R_{\underline{\mathbf{a}},k} := \sum_{j,i} \underline{\mathbf{a}}_j \gamma_k(j-i)\underline{\mathbf{a}}_i \geq 0$. As $R_{\underline{\mathbf{a}},k}$ is a finite linear combination of $\gamma_k(i)$ values the summation and limit operation commute, that is $\lim_{k\to\infty} R_{\underline{\mathbf{a}},k} = \sum_{j,i} \underline{\mathbf{a}}_j \gamma(j-i)\underline{\mathbf{a}}_i$. But as $R_{\underline{\mathbf{a}},k} \geq 0$ for all k its limit cannot be negative, proving the positive semi-definiteness of γ and hence the validity of ω.

A related question, that the domains of attraction of valid fixed points may contain 'invalid' functions which do not correspond to processes, remains open, and is not addressed here.

The different domains of attraction are already characterised precisely in terms of ϕ. In the remainder of this subsection we seek to describe them in more accessible terms. A fundamental realisation is that, because of theorem 4, definition 6 implies that *all* processes which have limit points are asymptotically self-similar, a fact which highlights the importance of the set of *SS* processes. Consider the domain of attraction of a given SS process. A natural question is whether it is non-trivial, that is if it contains more than just the fixed point itself. The issue remains open for the AP$_{q,c}$ family. In the case of the FN$_H$ family it is well known, except perhaps for the case $H = 0$, that they are non-trivial for all H. In the remainder of this section we help describe the domain of attraction of members of FN$_H$ with $H \leq 1/2$ by giving 'large' explicit subsets of them, including for $H = 0$. Section 5 provides examples for $H > 1/2$.

The processes which tend to FN$_H$ are characterised by:

$$\lim_{m\to\infty} \frac{\omega(nm)}{\omega(m)} = \lim_{m\to\infty} \frac{\phi(nm)}{\phi(m)} = n^{2H}, \text{ with } H \in [0,1]. \tag{14}$$

The FN$_H$ with $H = 1/2$ is just white noise, and we call its domain of attraction the set of *short-range dependent* processes. An important subclass is formed by processes for which $S_\infty = \lim_{n\to\infty} S(n) = \sum_{k=-\infty}^{\infty} \gamma(k)$ exists, and lies in $(0,\infty)$, the finite value indicating that the net influence of the past is not too heavy (essentially, any process whose spectral density exists and is finite but non-zero at the origin). For white noise $S_\infty = \mathcal{V}$. This subclass is quite large, for example it includes the frequently used autoregressive moving average (ARMA) models (3).

Recall the definition of asymptotic equivalence: $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) \to 1$.

Lemma 3. *If $S_\infty \in (0,\infty)$ is defined, then $\mathcal{V}^{(m)} \sim \frac{S_\infty}{m}$.*

A proof can be found in (8). We can now immediately give

Theorem 5. (Classic short-range dependence.)
If $S_\infty \in (0,\infty)$ is defined, then the process is in the domain of attraction of white noise.

Proof. From equation (4), property Æ–(ii) (see appendix), and lemma 3, it follows that $\omega(m) \sim mS_\infty$, and so $\lim_{m\to\infty} \phi(nm)/\phi(m) = n$. Thus members of the class are in the domain of attraction of white noise.

The fixed points with $H \in (0,1/2)$ typically receive little attention. We name the processes in their domains of attraction *constrained short-range dependent*. To justify this, we show that a large subclass of processes with $S_\infty = 0$ are in the domain of attraction. This means that the covariances conspire to exactly cancel, which can be thought of as a constraint.

Theorem 6. (Constrained short-range dependence.) *If $S_\infty = 0$ is defined, and $\gamma(k) \sim c_\gamma k^{2H-2}$ with $H \in (0, 1/2)$, then the process is in the domain of attraction of FN_H.*

The proof is left to the appendix as it uses results from the next section. That such processes exist can be justified using the frequency domain.

Finally we consider the $H = 0$ case, which has been largely ignored. The process FN_0 is very simple, with $\rho = \{1, -0.5, 0, 0 \ldots\}$, $S_\infty = 0$ and $\phi_\infty = \lim_{n\to\infty} \phi(n) = 1$. It is a moving average process of order one, and is also the first difference of white noise.

Theorem 7. (The DoA of FN_0 is large.) *Let X be any stationary process such that $\rho_\infty = \lim_{k\to\infty} \rho(k)$ exists and $\rho(1) < 1$. Then the differenced process $Y(i) = X(i+1) - X(i)$ is in the domain of attraction of FN_0.*

Proof. If we assume that $\phi_\infty \in (0, \infty)$ exists, then clearly $\lim_{m\to\infty} \phi(nm)/\phi(m) = 1$, corresponding to $H = 0$. Now by straightforward calculations one can show that $\phi_Y(m) = (1 - \rho(m))/(1 - \rho(1))$. Hence $\phi_\infty < \infty$ exists, and since this implies, using lemma 5 from the appendix, that $\phi_\infty \geq 1/4 > 0$, Y is in the domain of attraction of FN_0.

5. Discrete Regular Variation and Long-Range Dependence

Second-order self-similarity and long-range dependence (LRD) are usually thought of as related fields. In fact LRD time series constitute an important subset of *ASS* processes, characterised by an aggregated variance $\mathcal{V}^{(m)}$ which decays more slowly than m^{-1}, indicating 'long memory'. Despite their importance and widespread use however, there are a number of common misconceptions about the definition and properties of discrete LRD processes in the applied literature. One aim of this section is to clearly state and resolve these issues. We accordingly provide a complete treatment for some important and oft-quoted results for which correct proofs are nonetheless difficult or impossible to find.

LRD is related to the asymptotic properties of the autocovariance function, which is often taken to be *regularly varying*. It is not commonly appreciated however that regular variation in the discrete context does not enjoy all the properties of its well developed continuous parent (2). Our second aim is to clearly define the discrete form, state and prove its relevant properties, and clarify its relation to LRD and *ASS* in general. In this paper we restrict ourselves to second-order processes, although broader treatments of LRD are possible, for example (8).

5.1. Discrete regular variation

We must first define the standard concept of 'continuous' regular variation (2; 5).

Definition 7. (*Continuous regular variation (CRV).*) A function $\widetilde{f}$ defined on $\mathbb{R}^+$ is *regularly varying* at infinity with index α if

$$\lim_{t\to\infty} \frac{\widetilde{f}(tx)}{\widetilde{f}(t)} = x^\alpha, \quad \alpha \in \mathbb{R} \tag{15}$$

for every $x \in \mathbb{R}^+$ (it is sufficient that (15) be satisfied on a dense subset of $\mathbb{R}^+$, (5) page 275). If $\alpha = 0$ the function f is also said to be *slowly varying* (CSV).

Examples of slowly varying functions include any function with a positive finite limit at infinity, and any real power of $\ln x$. We ignore infinite α here.

Let CRV_α denote the set of regularly varying functions defined on $\mathbb{R}^+$ with index α. Some of their main properties are given in the following list.

CRV–(i) $\widetilde{f} \in \mathrm{CRV}_\alpha \Leftrightarrow \lim_{t\to\infty} \widetilde{f}(tx)/\widetilde{f}(t) = x^\alpha$, $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^+$ (Definition)

CRV–(ii) $\widetilde{f} \in \mathrm{CRV}_\alpha \Leftrightarrow \widetilde{f}(x) = \widetilde{s}(x)x^\alpha$, $\widetilde{s}(x) \in \mathrm{CSV}$ (alternative Definition)

CRV–(iii) $\widetilde{f} \in \mathrm{CRV}_\alpha$ and $\widetilde{g} \sim \widetilde{f} \Rightarrow \widetilde{g} \in \mathrm{CRV}_\alpha$

CRV–(iv) $\widetilde{f} \in \mathrm{CRV}_\alpha \Rightarrow \widetilde{f}(x) \sim \widetilde{f}(x + x_0)$, $\forall x_0$ constant.

The proof of property **CRV**–(ii) is straightforward, and **CRV**–(iii) is a direct consequence of **Æ**–(iv) and **Æ**–(ii). A proof of **CRV**–(iv) is given in the appendix.

Now consider a function $f(k)$, $k \in Z^+$. It is significant that in the literature on LRD where regular variation enters, there is rarely if ever any mention of a discrete definition – the properties of CRV are used without question. In fact, there is no discrete definition for which all the properties in the above list hold true. We propose the following one which preserves the majority of them in a natural way. Our choice agrees with the observations of (2) (page 52), and (6) on discrete regularly varying functions.

Definition 8. (*Discrete regular variation (DRV).*) A function f defined on $\mathbb{Z}^+$ is *regularly varying* at infinity with index α if there exists a $\widetilde{f} \in \mathrm{CRV}_\alpha$ such that $f(n) = \widetilde{f}(n)$ for all $n \in \mathbb{Z}^+$.

By defining DRV via sampling in this way, we guarantee that most of the properties of CRV are directly inherited. Of the above list, only **CRV**–(i) does not carry over exactly. To avoid any possibility of ambiguity, and to provide a convenient reference, we give the corresponding discrete list explicitly:

DRV–(i) $f \in \mathrm{DRV}_\alpha \Rightarrow \lim_{k\to\infty} f(kn)/f(k) = n^\alpha$, $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}^+$

DRV–(ii) $f \in \mathrm{DRV}_\alpha \Leftrightarrow f(k) = s(k)k^\alpha$, $s(k) \in \mathrm{DSV}$ (alternative Definition)

DRV–(iii) $f \in \mathrm{DRV}_\alpha$ and $g \sim f \Rightarrow g \in \mathrm{DRV}_\alpha$

DRV–(iv) $f \in \mathrm{DRV}_\alpha \Rightarrow f(k) \sim f(k + k_0)$, $\forall k_0$ constant.

Although these results are not difficult to demonstrate, we know of no reference for them. Proofs are therefore provided in the appendix.

The fact that property **DRV**–(i), in contrast to **CRV**–(i), only acts in one direction, shows that attempting to define DRV by analogy to equation (15) would not have been fruitful. Because k and n take only integer values, far fewer constraints are placed on the function f, allowing much wilder local behaviour. In contrast, equation (15) ensures that, in the limit, the *relative* size of local variations in $\widetilde{f}$ must be negligible.

Property **DRV**–(i) also plays a vital role in the understanding of *ASS*, as the limit on the right hand side is nothing other than the definition of a power-law type fixed point. Thus, if f were such a fixed point (with $\alpha \in [0, 2]$), the inability to infer that f is DRV clarifies the important fact that *regular variation has no a priori connection to asymptotic self-similarity*.

We can show now that the class of regularly varying autocovariance functions is invariant under (finite level) aggregation.

Theorem 8. (Persistence of power-law autocovariance functions.) *If an autocovariance function satisfies* $\gamma(k) = c_\gamma(k)k^\alpha$, $\alpha > 0$, *then for each fixed* $m \in \mathbb{Z}^+$, $\gamma^{(m)}(k) \overset{k}{\sim} (c_\gamma(k)m^\alpha)k^\alpha$, *where* $\overset{k}{\sim}$ *denotes asymptotic equivalence as* $k \to \infty$.

Proof. Take equation (3). According to **DRV**–(iv) we can replace each $\gamma(km \pm m \pm i)$ term by $\gamma(km)$. Since m is fixed, so is the number of terms in the summation, and using **Æ**–(iii) we have

$$\begin{aligned}\gamma^{(m)}(k) &\overset{k}{\sim} \frac{1}{m^2}\left[m\gamma(km) + 2\sum_{i=1}^{m-1} i\gamma(mk)\right] = \frac{1}{m^2}\left[m\gamma(km) + m(m-1)\gamma(mk)\right] \\ &= \gamma(mk) = c_\gamma(mk)m^{2H-2}k^{2H-2} \overset{k}{\sim} \left(c_\gamma(k)m^{2H-2}\right)k^{2H-2}.\end{aligned}$$

Another example of the utility of the discrete definition is that cumulative sums of functions with DRV can be conveniently derived from the corresponding continuous results.

Theorem 9. (Integration of regularly varying functions.) *Let $\widetilde{Z}(t) \in CRV_\alpha$, and let $\widetilde{L}(t)$ and $\widetilde{U}(t)$ be defined as:*

$$\widetilde{L}(t) := \int_0^t \widetilde{Z}(y)\,dy \qquad\qquad \widetilde{U}(t) := \int_t^\infty \widetilde{Z}(y)\,dy.$$

$$\text{(a) If } \alpha \geq -1 \quad \text{then} \quad \frac{t\widetilde{Z}(t)}{\widetilde{L}(t)} \to (1+\alpha), \;\; \text{and } \widetilde{L} \in RV_{\alpha+1}.$$

$$\text{(b) If } \alpha < -1 \quad \text{then} \quad \frac{t\widetilde{Z}(t)}{\widetilde{U}(t)} \to -(1+\alpha), \;\; \text{and } \widetilde{U} \in RV_{\alpha+1}.$$

A proof can be found in (13) page 17, and also (5), pages 279–281.

Theorem 10. (Summing discrete regularly varying functions.) *Let $K(n) \in DRV_\alpha$, and let $L(t)$ and $U(t)$ be defined as*

$$L(m) := \sum_{n=0}^{m-1} K(n), \qquad\qquad U(m) := \sum_{n=m}^{\infty} K(n).$$

$$\text{(a) If } \alpha \geq -1 \quad \text{then} \quad \frac{mK(m)}{L(m)} \to (1+\alpha), \;\; \text{and } L \in RV_{\alpha+1}.$$

$$\text{(b) If } \alpha < -1 \quad \text{then} \quad \frac{mK(m)}{U(m)} \to -(1+\alpha), \;\; \text{and } U \in RV_{\alpha+1}.$$

The proof is left to the appendix.

The two theorems above essentially tell us that regular variation is preserved under integration. Unfortunately the same is not true when differencing a function or sequence. Further examples of the use and implications of DRV appear in the next subsection.

5.2. Long-Range Dependence

In any reasonable sense, processes in the FN_H subclass with $H \in (0.5, 1]$ are long-range dependent. In keeping with our aggregation based approach to processes, a very broad and natural definition for LRD is therefore given by

Definition 9. (*LRD.*) Long-range dependent processes are those in the union of the domains of attraction of FN_H with $H \in (0.5, 1]$.

This definition will now be compared to the following alternatives from the literature.

Definition 10. (*LRD1.*) LRD1 processes are those whose autocovariance functions obey $\gamma(k) \sim c_\gamma k^{2H-2}$, $H \in (0.5, 1)$, $c_\gamma \in \mathbb{R}^+$.

This definition is the simplest and most frequently encountered. For example it is used in (4; 1).

Definition 11. (*LRD2.*) LRD2 processes are those whose autocovariance functions obey $\gamma(k) = c_\gamma(k) k^{2H-2}$, $H \in (0.5, 1)$, $c_\gamma \in$ DSV.

This choice generalises LRD1 in a natural way, by replacing the constant c_γ, a particular slowly varying function, with a general DSV function. It often appears with $\sim$ replacing $=$, but according to **DRV**–(iii) this is superfluous. In the literature LRD2 is often introduced but then immediately specialised to LRD1 (eg. (4)).

A number of additional 'alternatives' to LRD2 can be generated by using different, and we would argue misleading, definitions of DRV. We choose not to discuss such cases.

Definition 12. (*LRD3.*) LRD3 processes are those whose covariance sums obey $S_\infty = \lim_{i\to\infty} S(i) = \infty$.

This definition, used for example in (14), nicely captures the idea of LRD, being when the sum of the past has a strong impact.

Since LRD1 $\subset$ LRD2 we will not discuss LRD1 further. The remainder of the section deals with the nature of LRD2 and LRD3, their relationship to our definition of LRD, and related results.

The following four properties will be proved and discussed. Each of the last three provide classes of examples of LRD processes, complementing the results of section 4.2.

LRD–(i) For LRD2 processes, $\mathcal{V}^{(m)} \sim \frac{c_\gamma(m) m^{2H-2}}{H(2H-1)}$.

LRD–(ii) LRD2 $\subseteq$ LRD.

LRD–(iii) LRD2 $\neq$ LRD.

LRD–(iv) LRD2 $\subseteq$ LRD3.

*Proof of property **LRD**–(i).* Define $S_1(i) := \sum_{k=0}^{i-1} \gamma(k)$. According to theorem 10, $S_1 \in \mathrm{DRV}_{2H-1}$ and $S_1(i) \sim \frac{c_\gamma(i)}{2H-1} i^{2H-1}$. It is easy to see that because $2S_1(i) \to \infty$, $2S_1(i) \sim 2S_1(i) - \mathcal{V} + \gamma(i) = S(i)$. Since Æ is transitive (see appendix), we have

$$S(i) \sim 2\frac{c_\gamma(i)}{2H-1} i^{2H-1},$$

which by **DRV**–(iii) implies $S(i) \in \mathrm{DRV}_{2H-1}$. Now applying theorem 10 again for S, we obtain $w(m) = \sum_{i=0}^{m-1} S(i) \sim \frac{c_\gamma(m)}{H(2H-1)} m^{2H}$, and because of equation (4) and **Æ**–(ii):

$$\mathcal{V}^{(m)} \sim \frac{c_\gamma(m)}{H(2H-1)} m^{2H-2},$$

so $\mathcal{V}^{(m)}$ is also regularly varying with index $2H-2$.

This result appears in surprisingly few places, as typically the slowly varying prefactor, $\frac{c_\gamma(m)}{H(2H-1)}$, is omitted. One finds $\mathcal{V}^{(m)} \sim m^{2H-2}$ (implying an abuse of the Æ notation). Knowledge of the slowly varying part is however essential for purposes such as the determination of confidence intervals for mean estimates. The result for the special case of $c_\gamma(k)$ a constant is stated in (1), although not proved there, and in many places it is given with the prefactor omitted. Often (4) is cited in lieu of proof, although no formal proof appears there. Invalid proofs, and proofs relying on unproved lemmas, are not difficult to find.

*Proof of property **LRD**–(ii).* In the proof of **LRD**–(i) it was shown that $w \in RV_{2H}$ for processes in LRD2, so by **DRV**–(i)

$$\lim_{m\to\infty} \frac{\phi(nm)}{\phi(m)} = \lim_{m\to\infty} \frac{w(nm)}{w(m)} = n^{2H}$$

for every $n \in \mathbb{Z}^+$, so that ϕ is in the domain of attraction of FN$_H$.

This result is not surprising in view of theorem 8. The simplicity of the proof however is owed to the natural and complementary nature of the definitions of LRD and of DRV, together with the directness of the CTF approach, based on ϕ. Proofs in the literature are typically based on the ACF approach to asymptotic self-similarity.

*Proof of property **LRD**–(iii).* To show that LRD2 is a strict subset of LRD, a process will be constructed that is a member of LRD$\backslash$LRD2. Let X_1 and X_2 be independent copies of an FN$_H$ process with autocovariance function γ^*, $H \in (0.5, 1]$. We define $Y(t)$, $t \in \mathbb{Z}$ by deterministically alternating between the two copies:

$$Y(t) := \begin{cases} X_1(t/2), & t \text{ even} \\ X_2(\frac{t-1}{2}), & t \text{ odd} \end{cases}$$

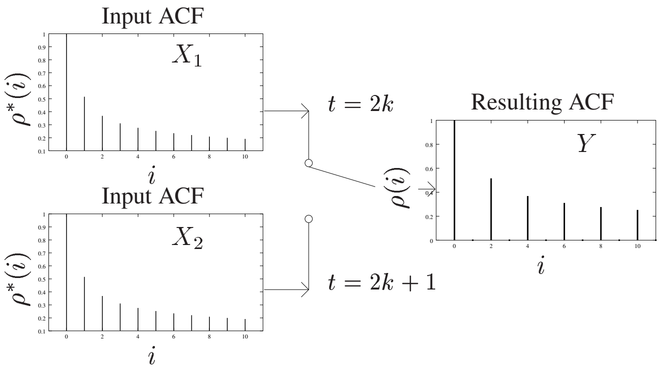

FIGURE 2: Example of a process in LRD$\backslash$LRD2. Odd lags compare different processes and are uncorrelated.

as illustrated in figure 2. It is not difficult to see that Y is a second-order stationary process, with autocovariance function

$$\gamma_Y(k) = \begin{cases} \gamma^*(k/2), & k \text{ even} \\ 0, & k \text{ odd} \end{cases},$$

and so

$$\omega_Y(m) = \begin{cases} 2\omega^*(m/2), & m \text{ even} \\ 2\omega^*((m-1)/2) + S^*((m-1)/2), & m \text{ odd} \end{cases} . \tag{16}$$

If m is even, then it follows trivially that equation (14) is satisfied for ω_Y for any n. If m is odd, then recalling that $\omega(m) = \omega(m-1) + S(m-1)$ and exploiting the monotonicity of ω^*, the asymptotically equivalent upper and lower bounds of $\frac{\omega_Y(nm)}{\omega_Y(m)}$ given in

$$\underbrace{\frac{\omega^*(n\frac{m-1}{2})}{\omega^*(\frac{m-1}{2})}}_{\substack{\downarrow \\ n^{2H}}} \underbrace{\frac{\omega^*(\frac{m-1}{2})}{\omega^*(\frac{m+1}{2})}}_{\substack{\downarrow \\ 1}} = \frac{\omega^*(n\frac{m-1}{2})}{\omega^*(\frac{m+1}{2})} \le \underbrace{\frac{\omega_Y(nm)}{\omega_Y(m)}}_{\substack{\downarrow \\ n^{2H}}} \le \frac{\omega^*(n\frac{m+1}{2})}{\omega^*(\frac{m-1}{2})} = \underbrace{\frac{\omega^*(\frac{m+1}{2})}{\omega^*(\frac{m-1}{2})}}_{\substack{\downarrow \\ 1}} = \underbrace{\frac{\omega^*(n\frac{m+1}{2})}{\omega^*(\frac{m+1}{2})}}_{\substack{\downarrow \\ n^{2H}}}$$

confirm that equation (14) is again satisfied.

In this example the monotonicity of the CTF of FN_H was exploited. With a slightly more sophisticated proof any LRD process could have been selected, showing that the set LRD$\setminus$LRD2 is quite large.

The question of whether LRD2 is a strict subset of LRD is not usually asked explicitly, however the opposite is sometimes implicitly assumed. Again, this can be linked to the misconception that **CRV**–(i) applies, and/or an unjustified differencing of a regularly varying ϕ.

The final definition, LRD3, raises new issues, as using the value of S_∞ to define a class can be viewed as a shift from a pointwise definition of convergence under aggregation, to one based on integrals of γ, of 'weak' convergence. We first offer a simple result on the behaviour of this alternative.

Theorem 11. *Assume S_∞ exists. If $S_\infty < \infty$ then $S_\infty^{(m)} = \frac{S_\infty}{m}$. If $S_\infty = \infty$ then $S_\infty^{(m)} = \infty$.*

Proof. Express $S^{(m)}$ in terms of S as $S^{(m)}(n) = \sum_{k=-n}^{n} \gamma^{(m)}(k) = \frac{1}{m^2}\sum_{i=mn}^{mn+m-1} S(i)$. From here the statement easily follows.

It follows that the three classes of S_∞ zero, finite non-zero, or infinite, are invariant under (finite level) aggregation. We now prove property **LRD**–(iii).

*Proof of property **LRD**–(iv).* From the proof of property **LRD**–(ii), $S_\infty = \infty$ for processes in LRD2, hence LRD2 $\subseteq$ LRD3.

It is tempting to conclude that, since by theorem 11 $S_\infty = \infty$ is preserved under aggregation, and since $S_\infty = \infty$ for each of the LRD fixed points, but no others, that LRD3 $\supseteq$ LRD, i.e. that every LRD process has infinite covariance sum. Similarly it may seem obvious that no process with $S_\infty = 0$ can be LRD, or that no short-range dependent process can have $S_\infty = \infty$. However none of these statements necessarily hold, as the pointwise convergence of $\rho^{(m)}$ does not control the finiteness or otherwise of the $S_\infty^{(m)}$. Not surprisingly however, under additional conditions such as regularly varying autocovariance function, they can be made to hold.

Appendix A. Asymptotic Equivalence and Regular Variation

A.1. Properties of asymptotic equivalence

Recall $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) \to 1$. In the following we will assume that $f_i(x) \sim g_i(x)$ for all i.

Æ–(i) Æ is symmetric ($f \sim g \Leftrightarrow g \sim f$), transitive ($f \sim g, g \sim h \Rightarrow f \sim h$), and reflexive ($f \sim f$).

Æ–(ii) $f_1(x)f_2(x) \sim g_1(x)g_2(x)$.

Æ–(iii) If $\alpha_i > 0, i < \infty$, constant then $\sum_i \alpha_i f_i(x) \sim \sum_i \alpha_i g_i(x)$ if all the f_i, g_i functions have the same sign near ∞.

Æ–(iv) $f_1(Cx) \overset{x}{\sim} g_1(Cx)$, where C is a positive constant.

Only **Æ**–(iii) will be proved as the others are obvious.

Proof of Æ–(iii). We first show that if f_1 and f_2 have the same sign near ∞ then $(f_1+f_2) \sim (g_1 + g_2)$. As $f_1 \sim g_1$ and $f_2 \sim g_2$, for all $\epsilon > 0$ there exists a x_0 such that for all $x > x_0$, $f_1(x) < (1+\epsilon)g_1(x)$, and $f_2(x) < (1+\epsilon)g_2(x)$, and so $(f_1+g_1)(x) < (1+\epsilon)(f_2+g_2)(x)$. Similarly, there exists x_1 such that for all $x > x_1$ $(f_1+g_1)(x) > (1-\epsilon)(f_2+g_2)(x)$, which means that $(f_1+f_2) \sim (g_1+g_2)$. Now using **Æ**–(ii) with $f_2(x) \equiv g_2(x) \equiv C$ the result follows from induction.

A.2. Properties of continuous regular variation

CRV–(iv) $\widetilde{f} \in \mathrm{CRV}_\alpha \Rightarrow \widetilde{f}(x) \sim \widetilde{f}(x+x_0)$, $\forall x_0$ constant.

Proof. In (5) page 282 and (13) page 35 it was shown that if $\widetilde{s} \in \mathrm{SV}$ then it has an integral representation of the form:

$$\widetilde{s}(x) = a(x)e^{\left(\int_1^x \frac{\epsilon(y)}{y}dy\right)}, \tag{17}$$

where $\epsilon(x) \to 0$ and $a(x) \to c$, $0 < c < \infty$ as $x \to \infty$. Using this representation

$$\frac{\widetilde{f}(x+x_0)}{\widetilde{f}(x)} = \left(\frac{x+x_0}{x}\right)^\alpha \frac{a(x+x_0)}{a(x)} \exp\left(\int_x^{x+x_0} \frac{\epsilon(y)}{y}dy\right).$$

We know that $\left(\frac{x+x_0}{x}\right)^\alpha \to 1$, $\frac{a(x+x_0)}{a(x)} \to 1$ because $a(x) \to c$, $0 < c < \infty$. We also have that $\int_x^{x+x_0} \frac{\epsilon(y)}{y}dy \to 0$ because $\epsilon(y) \to 0$.

A.3. Properties of discrete regular variation

Properties **DRV**–(i) and **DRV**–(iv) follow directly from their continuous equivalents, since subsequences of convergent functions converge to the same limit.

DRV–(ii) $f \in \mathrm{DRV}_\alpha \Leftrightarrow f(k) = s(k)k^\alpha$, $s(k) \in \mathrm{DSV}$ (alternative Definition)

Proof. ($\Leftarrow$) Since $s \in \mathrm{DSV}$, by definition there exists a function $\widetilde{s} \in \mathrm{CSV}$ s.t. $\widetilde{s}(k) = s(k)$, $\forall k \in Z^+$. Define $\widetilde{f}(x) := \widetilde{s}(x)x^\alpha$. According to **CRV**–(ii) $\widetilde{f} \in \mathrm{CRV}_\alpha$ and clearly $\widetilde{f}(k) = s(k)k^\alpha$. Hence $f \in \mathrm{DRV}_\alpha$ by definition. The proof of the opposite direction is very similar.

The following lemma will be used for the proof of **DRV**–(iii) and Theorem 10:

Lemma 4. (Step function interpolation.) *If $f \in DRV_\alpha$ then the continuous time function $\widetilde{f}_s(x) := f(\lfloor x \rfloor)$ is in CRV_α, where $\lfloor x \rfloor$ denotes the integer that is the closest to x but smaller in absolute value.*

Proof. Because $f \in \mathrm{DRV}_\alpha$ there exists a continuous function $\widetilde{f} \in \mathrm{CRV}_\alpha$ s.t. $\widetilde{f}(n) = f(n)$, $\forall n \in \mathbb{Z}^+$. If one can show that $\widetilde{f}(x) \sim \widetilde{f}(\lfloor x \rfloor) = \widetilde{f}_s(x)$, then according to **CRV**–(iii) $\widetilde{f}_s \in \mathrm{CRV}_\alpha$. We use the integral representation (17) of slowly varying functions, to obtain

$$\frac{\widetilde{f}(x)}{\widetilde{f}(\lfloor x \rfloor)} = \left(\frac{x}{\lfloor x \rfloor}\right)^\alpha \frac{a(x)}{a(\lfloor x \rfloor)} \exp\left(\int_{\lfloor x \rfloor}^x \frac{\epsilon(y)}{y} dy\right).$$

The first two terms on the right hand side converge to 1, and because the size of the interval $(\lfloor x \rfloor, x)$ is bounded and $\epsilon(x) \to 0$, the third term also converges to 1.

DRV–(iii) $f \in \mathrm{DRV}_\alpha$ and $g \sim f \Rightarrow g \in \mathrm{DRV}_\alpha$.

Proof. Let $\widetilde{f}_s(x) := f(\lfloor x \rfloor)$ and $\widetilde{g}_s(x) := g(\lfloor x \rfloor)$, $\forall x \in \mathbb{R}^+$. According to lemma 4 $\widetilde{f}_s(x) \in \mathrm{CRV}_\alpha$. It is clear that $\widetilde{f}_s \sim \widetilde{g}_s$, and so from **CRV**–(iii) $\widetilde{g}_s \in \mathrm{CRV}_\alpha$, and because $g(n) = \widetilde{g}_s(n)$, $\forall n \in \mathbb{Z}^+$ we have $g \in \mathrm{DRV}_\alpha$.

Theorem 12. (Summing regularly varying sequences, (Theorem 10).) *Let $K(n) \in DRV_\alpha$, and let $L(t)$ and $U(t)$ be defined as*

$$L(m) := \sum_{n=0}^{m-1} K(n), \qquad U(m) := \sum_{n=m}^{\infty} K(n).$$

(a) If $\alpha \geq -1$ then $\dfrac{mK(m)}{L(m)} \to (1+\alpha)$, and $L \in RV_{\alpha+1}$.

(b) If $\alpha < -1$ then $\dfrac{mK(m)}{U(m)} \to -(1+\alpha)$, and $U \in RV_{\alpha+1}$.

Proof. Define the continuous function $\widetilde{K}(x) := K(\lfloor x \rfloor)$. According to lemma 4 $\widetilde{K} \in RV_\alpha$, and so according to theorem 9

$$\frac{x\widetilde{K}(x)}{\int_0^x \widetilde{K}(y)dy} \to (1+\alpha) \qquad \text{or} \qquad \frac{x\widetilde{K}(x)}{\int_x^\infty \widetilde{K}(y)dy} \to -(1+\alpha)$$

depending on the value of α. Subsequences of convergent functions converge to the same limit so the above equation remains valid if $x \in [0, \infty)$ is replaced by $n = 0, 1, 2, \ldots$. But in this case we know that $\widetilde{K}(n) = K(n)$ and also that $\int_0^n \widetilde{K}(y)dy = \sum_{i=0}^{n-1} K(n)$ and $\int_n^\infty \widetilde{K}(y)dy = \sum_{i=n}^\infty K(n)$. So it follows that

$$\frac{nK(n)}{\sum_{i=0}^{n-1} K(n)} \to (1+\alpha) \qquad \text{or} \qquad \frac{nK(n)}{\sum_{i=n}^{\infty} K(n)} \to -(1+\alpha)$$

depending again on the value of α. Because L and U are the sampled equivalents of $\widetilde{L}$ and $\widetilde{U}$ of theorem 9 respectively they are both regular varying.

Appendix B. Fixed Points, Positive Semi-Definiteness, and Domains of Attraction.

CTF–(i) $\phi(m-n) \leq 2(\phi(m)+\phi(n))$ for all $m, n \in \mathbb{Z}^+$, $m-n \geq 1$.

Proof. The statement is equivalent to $\omega(m-n) \leq 2(\omega(m)+\omega(n))$. Introduce the sums $A := \sum_{i=1}^{m} X(i)$ and $B := \sum_{i=m-n+1}^{m} X(i)$. It follows that $\omega(m-n) = E[(A-B)^2] = E[A^2] + E[B^2] - 2E[AB]$, $\omega(m) = E[A^2]$, $\omega(n) = E[B^2]$. Using $E[(A+B)^2]$ and combining, the result quickly follows.

CTF–(ii) For any $n \in \mathbb{Z}^+$, $|S(n)/\mathcal{V}| < 2\sqrt{\phi(n)} + C$, where C is a constant independent of n.

Proof. Define the symmetric extension of ϕ as $\phi(-m) = \phi(m)$, with $\phi(0) = 0$. Define $s(n) = S(n)/\mathcal{V}$, and extent it to all $n \in \mathbb{Z}$ via $s(n) = \phi(n+1) - \phi(n)$. For all $n > 0$, $k \in \mathbb{Z}$ we can now define

$$b = \phi(n+k) - \phi(n) - ks(n).$$

For $k > 0$, $\phi(n+k) = \phi(n) + \sum_{i=n}^{n+k-1} s(i)$, and so $b = \sum_{i=n}^{n+k-1}(s(i) - s(n))$. Since $|\rho(j)| \leq 1$ for any j, $|s(i) - s(n)| \leq 2(i-n)$, and using the triangle inequality we obtain $|b| \leq |k||k-1|$. If $k < 0$ then $b = \sum_{i=-1}^{k}(s(n) - s(n+i))$ and similarly one can show that $|b| \leq |k||k-1|$ for all $k \in \mathbb{Z}$. It follows that, regardless of the value of n,

$$0 \leq \phi(n+k) \leq \phi(n) + ks(n) + k^2 + |k|, \text{ for any } k \in Z.$$

Setting $k := -\lfloor s(n)/2 \rfloor$ it then follows that

$$0 \leq \phi(n) - \frac{s(n)^2}{2} + \frac{|s(n)|}{2} + \frac{s(n)^2}{4} + \frac{|s(n)|}{2} = \phi(n) - \frac{s(n)^2}{4} + |s(n)|. \qquad (18)$$

The largest possible $d := |s(n)|$ that satisfies (18) for a given $\phi(n)$ is the larger solution of the quadratic equation

$$\phi(n) - \frac{d^2}{4} + d,$$

that is $|s(n)| \leq |s(n)|_{\text{max}} = 8 + \sqrt{1 + 4\phi(n)} \leq 2\sqrt{\phi(n)} + 9$. Choosing any $C > 9$ completes the proof.

Lemma 5. (Most valid ϕ are eventually bounded above zero..) *Given a positive semi-definite γ with $\mathcal{V} > 0$, if* $\liminf_{m\to\infty} S(m) \geq 0$, *then* $\liminf_{m\to\infty} \phi(m) \geq 1/4$.

Proof. Let Γ_m be a $m \times m$ positive semi-definite covariance matrix: $[\Gamma_m]_{i,j} = \text{Cov}[X(i), X(j)]$, $\mathbf{1}$ be the identity vector, and $\mathbf{e} = \{0, 0, \cdots, 0, \epsilon\}^T$ where ϵ is positive. Letting $\mathbf{a} = \mathbf{1} - \mathbf{e}$ we have

$$\begin{aligned} \mathbf{a}^T \Gamma_m \mathbf{a} &= \mathbf{1}^T \Gamma_m \mathbf{1} - \mathbf{e}^T \Gamma_m \mathbf{1} - \mathbf{1}^T \Gamma_m \mathbf{e} + \mathbf{e}^T \Gamma_m \mathbf{e} \\ &= w(m) - \epsilon[\mathcal{V} + S(m-1)] + \epsilon^2 \mathcal{V} \ \geq 0 \qquad (19) \end{aligned}$$

by positive semi-definiteness. Since for m sufficiently large $S(m-1)$ is either positive, or negative but arbitrarily small, $w(m)$ becomes lower bounded in the limit by $\mathcal{V}\epsilon(1-\epsilon)$. As this quadratic in ϵ is maximised at $\epsilon = 1/2$, we have $\liminf_{m\to\infty} w(m) \geq \mathcal{V}/4$ from which the result follows.

Theorem 13. (Constrained short-range dependence (Theorem 6).) *If* $S_\infty = 0$ *is defined, and* $\gamma(k) \sim c_\gamma k^{2H-2}$ *with* $H \in (0, 1/2)$*, then the process is in the domain of attraction of* FN_H .

Proof. It will be shown that $\omega \in RV_{2H}$, which according to **DRV**–(i) and (14) yields the result.

Define the function $S_1^-(n) := \sum_{k=n}^{\infty} \gamma(k)$. We know that $\gamma \in RV_\alpha$ with $\alpha < -1$, so according to theorem 10b

$$S_1^-(n) \sim -\frac{c_\gamma(n)}{2H-1} n^{2H-1} \in RV_{2H-1}.$$

It will now be shown how to express $\omega(m)$ in terms of S_1^-. $0 = S_\infty = S(n) + 2\sum_{k=n+1}^{\infty} \gamma(k)$ and so $S(n) = -2\sum_{k=n+1}^{\infty} \gamma(k) = -2S_1^-(n+1)$. Hence

$$\omega(m) = \sum_{n=0}^{m-1} S(n) = -2\sum_{n=0}^{m-1} S_1^-(n+1).$$

We know that according to **DRV**–(iv) $S_1^-(n+1) \sim S_1^-(n)$ and so according to **DRV**–(iii) $S_1^-(n+1) \in RV_{2H-1}$ and so theorem 10 can be used to show that

$$\omega(m) \sim \frac{c_\gamma(m)}{H(2H-1)} m^{2H} \in RV_{2H}.$$

as required, since $\phi(m) = \omega(m)/\omega(1)$.

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References

[1] BERAN, J. (1994). *Statistics for Long-Memory Processes*. Chapman and Hall, New York.

[2] BINGHAM, N., GOLDIE, C. AND TEUGELS, J. (1987). *Regular Variation*. Cambridge University Press, Cambridge England.

[3] BROCKWELL, P. AND DAVIS, R. (1996). *Time Series: Theory and Methods*. Springer.

[4] COX, D. R. (1984). *in Long-range dependence: a review,* H.A. David and H.T. David editors. Iowa State University Press, Ames (IA). ch. Statistics : an Appraisal, pp. 55–74.

[5] FELLER, W. (1970). *An Introduction to Probability Theory and its Applications* second ed. vol. II. John Wiley and Sons, Brisbane.

[6] GALAMBOS, J. AND SENETA, E. (1973). Regularly Varying Sequences. *Proceedings of the American Mathematical Society* **41,** 110–116.

[7] GEFFERTH, A., VEITCH, D., RUZSA, I., MARICZA, I. AND MOLNÁR, S. (2002). A new class of second order self-similar processes. *submitted for publication*.

[8] HEYDE, C. C. AND YANG, Y. (1997). On Defining Long-Range Dependence. *Journal of Applied Probability* **34,** 939–944.

[9] LAMPERTI, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc* **104,** 62–78.

[10] MAJOR, P. (1981). *Multiple Wiener-Itô Integrals* vol. 849 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, New York.

[11] NARKIEWICZ, W. (1974). *Elementary and analytic theory of algebraic numbers*. PWN, Warsaw.

[12] NATHANSON, M. B. (1996). *Additive Number Theory. The Classical Bases*. Graduate Texts in Mathematics 164. Springer.

[13] RESNICK, S. I. (1987). *Extreme Values, Regular Variation and Point Processes*. Springer, New York.

[14] SAMORODNITSKY, G. AND TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman and Hall.

[15] SINAI, Y. G. (1976). Self-Similar Probability Distributions. *Theory of Probability and its Applications* **21,** 64–80.

[16] VERVAAT, W. (1987). Properties of General Self-Similar Processes. *Bull. Int. Stat. Institute* **52,** 199–216.